Rough Surfaces in the sense of Alexandrov and Reshetynak

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Contents

1 Introduction

Classical differential geometry is built on the assumption of smoothness. Curvature is traditionally defined using the second derivatives of a smooth metric. This approach breaks down once we encounter a wider range of surfaces, objects with sharp corners such as polyhedral surfaces for example. Does there exist a theory that bridges the notions we have from classical smooth geometry to a broader framework that encompasses a much wider class of surfaces?

This was a question that was heavily studied in the mid 20th century by the Russian mathematician A.D. Alexandrov who developed an axiomatic approach to defining surfaces with an intrinsic metric for which we reach a notion of curvature. In the 60s another Russian mathematician, Y.G. Reshetynak approached the question analytically.

The main focus of this project is to establish a remarkable connection between these two seemingly different approaches:

1. Alexandrov's Synthetic Approach: A geometric method that defined curvature via triangle comparisons in length spaces.
2. Reshetynak's Analytic Approach: A theory of distances induced by the difference between two subharmonic functions.

We will show that these two approaches lead to the same class of surfaces, those of bounded integral curvature. By establishing this link, we can apply a wider range of analytical tools to study singular geometric structures.

2 Riemannian Geometry

2.1 Preliminaries

Definition 2.1. A smooth n-dimensional manifold is a Hausdorff, 2^{nd} countable topological space with an open cover $\{U_\alpha\}_{\alpha\in\mathcal{I}}$ and homeomorphisms $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ such that the transition functions

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

are C^∞ for all $\alpha, \beta \in \mathcal{I}$. Each ϕ_α is called a smooth chart and the collection $\{(U_\alpha, \phi_\alpha)\}$ is called a smooth atlas of charts.

Definition 2.2. Let M be a manifold. We define a Riemannian metric g as a $\binom{2}{0}$-tensor field such that, for all $p \in M$, g_p is positive definite and symmetric.

The pair (M, g) is called a Riemannian manifold if g is a Riemannian metric on M. If M has dimension 2 then we call (M, g) a Riemannian surface.

We currently possess a local structure, since at each $p \in M$, the metric endows T_pM with an inner product. This allows us to make sense of geodesics, paths of zero acceleration. These can be thought of as analogues of straight lines. Seeing as these are dictated locally by the metric, can we get a complete picture of the manifold as a whole? This leads us to a global property, completeness.

Definition 2.3. A Riemannian manifold is said to be (geodesically) complete if every maximal geodesic $\gamma : I \to M$ is defined for all time $t \in \mathbb{R}$.

Consider $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ with the round metric for $\mathbb{R}^3$.

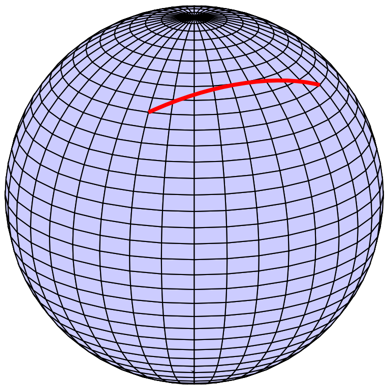

Figure 1: Geodesic between 2 points.

A short calculation shows that all geodesics on S^2 are arcs of the great circles.

Parameterised by

$$r(t) = (\cos(t))\mathbf{p} + (\sin(t))\mathbf{v}.$$

Where $\mathbf{p} \in S^2$ and unit vector $\mathbf{v} \in T_{\mathbf{p}}S^2$. It is clear that this is defined for all $t \in \mathbb{R}$ and so we see that S^2 is geodesically complete.

Now consider the upper half plane $\mathbb{H} = \{(x, y) \in R^2 \mid y > 0\}$ with two different metrics, the Euclidean metric $g_{\text{Euc}} = dx^2 + dy^2$ and the hyperbolic metric $g_{\text{hyp}} = \frac{dx^2+dy^2}{y^2}$.

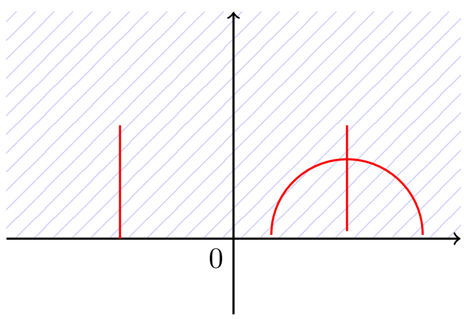

Figure 2: Geodesics on $\mathbb{H}$. Left hand side with g_{Euc}, right with g_{hyp}.

Under g_{Euc} our geodesics are simply straight lines. Let $\gamma(t) = (-1, 1 - t)$. It is clear that this geodesic cannot be extended beyond $t = 1$ within $\mathbb{H}$. Thus $(\mathbb{H}, g_{\text{Euc}})$ is geodesically incomplete.

To understand the geodesics under g_{hyp} we solve the geodesic equation. Note that the Christoffel symbols are as follows:

$$\Gamma^x_{xy} = \Gamma^x_{yx} = \Gamma^y_{yy} = -\frac{1}{y},\ \Gamma^y_{xx} = \frac{1}{y},\ \Gamma^x_{xx} = \Gamma^x_{yy} = \Gamma^y_{xy} = \Gamma^y_{yx} = 0.$$

The geodesics are thus the curves that solve the following system of ODEs:

$$\frac{d^2x}{dt^2} - \frac{2}{y}\frac{dx}{dt}\frac{dy}{dt} = 0, \qquad \frac{d^2y}{dt^2} + \frac{1}{y}\left[\left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2\right] = 0.$$

Computing the following derivative and substituting the above expressions for the second derivatives yields

$$\frac{d}{dt}\left(y\frac{dy}{dt}\left(\frac{dx}{dt}\right)^{-1} + x\right) = \left(\frac{dy}{dt}\right)^2\left(\frac{dx}{dt}\right)^{-1} + y\frac{d^2y}{dt^2}\left(\frac{dx}{dt}\right)^{-1} - y\frac{dy}{dt}\frac{d^2x}{dt^2}\left(\frac{dx}{dt}\right)^{-2} + \frac{dx}{dt} = 0.$$

It follows that either $\frac{dx}{dt} = 0$ or $y\frac{dy}{dt} + x\frac{dx}{dt} = A\frac{dx}{dt}$ for some constant A. We can conclude that geodesics are the straight vertical lines and the circles with centre on the line $y = 0$.

For a vertical path along $x = 1$ we see that the hyperbolic distance is $d_{\text{hyp}} = \int_{y_2}^{y_1}\frac{|dy|}{y} = \ln\left(\frac{y_1}{y_2}\right)$. So from $y = 1$ to $y = \varepsilon$ we get that $d_{\text{hyp}} = -\ln(\varepsilon)$. As $\varepsilon \to 0$, $d_{\text{hyp}} \to \infty$. A similar but more involved computation yields the same for geodesics along circles. So as seen in Figure 2, geodesics cannot reach the boundary in finite time and hence are defined for all $t \in \mathbb{R}$. We have shown that $(\mathbb{H}, g_{\text{hyp}})$ is geodesically complete.

Definition 2.4. Suppose g_1 and g_2 are two Riemannian metrics on M, we say they are conformally equivalent if there exists a smooth function $f : M \to (0, \infty)$ such that $g_1 = fg_2$.

This is an equivalence relation with the equivalence class $[g]$ being called a conformal class. In particular it encodes the notion of angles on the surface. A choice of conformal class on M is the conformal structure.

Definition 2.5. Local coordinates (u, v) on $U \subset M$ are said to be isothermal if the metric can be expressed as

$$g = \rho(\mathrm{d}u^2 + \mathrm{d}v^2),$$

where ρ is a positive smooth function on U called the conformal factor.

In these coordinates the metric differs from the Euclidean one only by the conformal factor. In other words, it is locally conformally flat.

Example 2.6. (Mercator's projection) Let $\mathbf{x}(\theta, \phi) = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$ be the standard spherical coordinates on S^2 where $\theta \in (0, \pi), \phi \in (0, 2\pi)$.

It can be seen that the induced metric is given by

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2.$$

We wish to find coordinates (u, v) such that definition 2.5 is satisfied. Due to the rotational symmetry of the sphere we make a general choice $u = u(\theta), v = \phi$. A substitution of $\mathrm{d}u = u'(\theta)\mathrm{d}\theta$ gives

$$\mathrm{d}s^2 = \frac{1}{(u'(\theta))^2}\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2.$$

Solving $u'(\theta) = \frac{1}{\sin(\theta)}$ yields $u(\theta) = \ln\tan\frac{\theta}{2}$. Where we have taken the convention of choosing $C = 0$ so that the equator $\theta = \frac{\pi}{2}$ maps to $u = \ln(1) = 0$. We can thus find a new parametrization given by

$$\mathbf{y}(u, v) = (\mathrm{sech}(u)\cos(v), \mathrm{sech}(u)\sin(v), \tanh(u))$$

It now follows by a direct computation that our new metric induced by $\mathbf{y}$ is of the form

$$\mathrm{d}s^2 = \mathrm{sech}^2(u)(\mathrm{d}u^2 + \mathrm{d}v^2).$$

Note that in both definitions 2.4 and 2.5 we are not specifying a dimension, they hold true for an arbitrary n. It turns out that when M has dimension 2 we can always find such coordinates for any Riemannian metric. This fact is not true in higher dimensions where we require some additonal conditions to assert this. We follow the set up in (Imayoshi & Taniguchi 2012, p. 20). Take a general metric given by

$$\mathrm{d}s^2 = E\,\mathrm{d}x^2 + 2F\,\mathrm{d}x\,\mathrm{d}y + G\,\mathrm{d}y^2$$

in local coordinates (x, y). Let us define $z = x + iy$ and $\bar{z} = x - iy$. In terms of the $\mathrm{d}z$ and $\mathrm{d}\bar{z}$, the metric can be expressed as

$$\mathrm{d}s^2 = \lambda\,|\mathrm{d}z + \mu\mathrm{d}\bar{z}|^2 \;=\; \lambda(\mathrm{d}z + \mu\mathrm{d}\bar{z})(\mathrm{d}\bar{z} + \bar{\mu}\mathrm{d}z). \tag{1}$$

Where the functions λ and μ can be computed to be

$$\lambda = \frac{1}{4}\left(E + G + 2\sqrt{EG - F^2}\right), \quad \mu = \frac{E - G + 2iF}{E + G + 2\sqrt{EG - F^2}}.$$

Take isothermal coordinates (u, v) as in definition 2.5 and set a new complex coordinate $w = u + iv$. So the metric looks like

$$ds^2 = \rho(u, v)(du^2 + dv^2).$$

The analogue to equation (1) which we find by computing λ, μ with $E = G = \rho$ and $F = 0$

would be

$$ds^2 = \rho(w, \overline{w})|dw|^2.$$

Additionally dw can be expressed as

$$\mathrm{d}w = \frac{\partial w}{\partial z}\mathrm{d}z + \frac{\partial w}{\partial \bar{z}}\mathrm{d}\bar{z}.$$

Substituting this into the metric equation yields

$$\mathrm{d}s^2 = \rho|\mathrm{d}w|^2 = \rho\left|\frac{\partial w}{\partial z}\mathrm{d}z + \frac{\partial w}{\partial \bar{z}}\mathrm{d}\bar{z}\right|^2 = \rho\left|\frac{\partial w}{\partial z}\right|^2\left|\mathrm{d}z + \frac{\frac{\partial w}{\partial \bar{z}}}{\frac{\partial w}{\partial z}}\mathrm{d}\bar{z}\right|^2.$$

It follows that an isothermal coordinate w for ds^2 exists if the following PDE has non trivial solutions.

$$\frac{\partial w}{\partial \overline{z}} = \mu\frac{\partial w}{\partial z}.$$

This is known as the Beltrami equation and such solutions can be proven to exist. We refer the reader to chapter 4.2 in (Imayoshi & Taniguchi 2012) for a full proof.

One nice feature of this particular set of coordinates is that it gives a nice form for the Gaussian curvature.

Lemma 2.7. *Let X be an orthogonal parametrization of some surface S. The Gaussian curvature can be computed to be*

$$K = -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\}.$$

Proof. Exercise in (Do Carmo 2016, p. 237). We calculate the Christoffel symbols as follows

$$\Gamma^1_{11} = \frac{E_u}{2E}, \qquad \Gamma^2_{11} = -\frac{E_v}{2G}, \qquad \Gamma^1_{12} = \frac{E_v}{2E}.$$

$$\Gamma^2_{12} = \frac{G_u}{2G}, \qquad \Gamma^1_{22} = -\frac{G_u}{2E}, \qquad \Gamma^2_{22} = \frac{G_v}{2G}.$$

The lemma now follows by the following computation

$$
\begin{aligned}
K &= -\frac{1}{E}\left\{\left(\Gamma_{12}^{2}\right)_u - (\Gamma_{11}^{2})_v + \Gamma_{12}^{1}\Gamma_{11}^{2} + \Gamma_{12}^{2}\Gamma_{12}^{2} - \Gamma_{11}^{2}\Gamma_{22}^{2} - \Gamma_{11}^{1}\Gamma_{12}^{2}\right\} \\
&= -\frac{1}{2\sqrt{EG}}\left\{-\frac{E_uG_u}{2E\sqrt{EG}} - \frac{E_v^2}{2E\sqrt{EG}} + \frac{2\sqrt{G}}{\sqrt{E}}\left(\frac{E_v}{2G}\right)_v + \frac{E_vG_v}{2G\sqrt{EG}}\right. \\
&\qquad \left. + \frac{G_u^2}{2G\sqrt{EG}} + \frac{2\sqrt{G}}{\sqrt{E}}\left(\frac{G_u}{2G}\right)_u\right\} \\
&= -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_{vv}}{\sqrt{EG}} - \frac{E_v(E_vG+EG_v)}{2EG\sqrt{EG}}\right) + \left(\frac{G_{uu}}{\sqrt{EG}} - \frac{G_u(E_uG+EG_u)}{2EG\sqrt{EG}}\right)\right\} \\
&= -\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\}.
\end{aligned}
$$

□

Theorem 2.8. *In isothermal coordinates, the formula for the Gaussian curvature K is given by*

$$K = -\frac{1}{2\rho}\Delta(\log\rho).$$

Proof. Using Lemma 2.7 in the case where $E = G = \rho(u,v)$ yields

$$
\begin{aligned}
K &= -\frac{1}{2\sqrt{\rho^2}}\left\{\left(\frac{\rho_v}{\rho}\right)_v + \left(\frac{\rho_u}{\rho}\right)_u\right\} = -\frac{1}{2\rho}\left(\log(\rho)_{v,v} + \log(\rho)_{u,u}\right) \\
&= -\frac{1}{2\rho}\Delta(\log\rho).
\end{aligned}
$$

□

2.2 From a smooth surface to a Riemann Surface

Definition 2.9. A Riemann surface X is a Hausdorff, 2^{nd} countable topological space equipped with an atlas of charts $\{(U_i,\phi_i)\}$ such that each $\phi_i : U_i \to V_i$ is a homeomorphism from an open set $U_i \subset X$ to an open set $V_i \subset \mathbb{C}$. For any two overlapping charts U_i and U_j, the transition map

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j),$$

is a holomorphic function between these two open subsets of the complex plane.

Since every manifold has a smooth Riemannian metric we get a conformal structure on our surface for free. However, some care needs to be had. A Riemann surface is an orientable surface [1], so if we wish to turn our surface into a Riemann surface, we must begin with an orientable surface and specify an orientation.

With this in mind, we now show that by use of isothermal coordinates we can upgrade our conformal structure to a complex one. We have already seen that around every point p we can find an open set which is conformally flat. Let us denote the complex coordinate z to be $x + iy$ where (x, y) is a positively oriented isothermal coordinate system. We would like to show that our transition maps are holomorphic. To this end, suppose we have two different isothermal coordinates around a point. $z = x + iy$ and $w = a + ib$. Expressing ∂_x, ∂_y in the other basis gives us

$$\frac{\partial}{\partial x} = a_x \frac{\partial}{\partial a} + b_x \frac{\partial}{\partial b}, \qquad \frac{\partial}{\partial y} = a_y \frac{\partial}{\partial a} + b_y \frac{\partial}{\partial b},$$

Since we have that both basis are orthogonal and have the same norm we have the following equalities:

$$a_x^2 + b_x^2 = a_y^2 + b_y^2. \tag{1}$$

$$a_x a_y + b_x b_y = 0. \tag{2}$$

Orientability (both charts positively oriented) implies the Jacobian determinant is positive:

$$a_x b_y - b_x a_y > 0. \tag{3}$$

From (2) the vectors (a_x, b_x) and (a_y, b_y) are orthogonal, (1) then tells us they have the same length. It follows that one is a $90°$ rotation of the other:

$$(a_y, b_y) = \pm(-b_x, a_x).$$

Substituting into (3) then gives us that we must have counterclockwise rotation, so

$$a_y = -b_x, \qquad b_y = a_x,$$

which are exactly the Cauchy–Riemann equations. Consequently, an oriented 2-dimensional Riemannian manifold is naturally a Riemann surface. The metric provides the conformal structure, and the isothermal coordinates provide the complex coordinates that make the transition functions holomorphic.

3 Length Spaces

Definition 3.1. Let (X, d) be a metric space. The length of a curve $\gamma : [a, b] \to X$ with respect to a metric d is defined as follows

$$L(\gamma) = \sup \sum_{i=0}^{n-1} d((\gamma(t_i), \gamma(t_{i+1})))$$

[1]Note that the determinant of the Jacobian of a holomorphic map is always non-negative so every transition map in a Riemann surface atlas is orientation-preserving.

where the supremum is over all partitions $\{t_0, t_1, \ldots, t_n\}$. If $L(\gamma) < \infty$, we say that γ is a rectifiable path in X.

Proposition 3.2. *Suppose $\{\gamma_i\}$ is a sequence of rectifiable paths in X such that*

$$\lim_{i\to\infty} \gamma_i(t) = \gamma(t)$$

for all t, for a rectifiable path γ. Then

$$L(\gamma) \leq \liminf_{t\to\infty} L(\gamma_i).$$

Proof. (Alexandrov 1948, p. 63) We begin by assuming that

$$S = \liminf_{i\to\infty} L(\gamma_i) < \infty.$$

Choose a subsequence γ_{i_k} such that $L(\gamma_{i_k}) \to S$ as $k \to \infty$. Let $a = t_0 < t_1 < \cdots < t_m = b$ be an arbitrary partition and $\varepsilon > 0$. By pointwise convergence at each t_j, there exists K such that for all $k \geq K$ and all j:

$$d(\gamma_{i_k}(t_j), \gamma(t_j)) \leq \frac{\varepsilon}{2m}.$$

It follows by the triangle inequality that,

$$d(\gamma(t_{j-1}), \gamma(t_j)) \leq d(\gamma_{i_k}(t_{j-1}), \gamma_{i_k}(t_j)) + \frac{\varepsilon}{m}.$$

Summing over j yields

$$\sum_{j=1}^{m} d(\gamma(t_{j-1}), \gamma(t_j)) \leq \sum_{j=1}^{m} d(\gamma_{i_k}(t_{j-1}), \gamma_{i_k}(t_j)) + \varepsilon \leq L(\gamma_{i_k}) + \varepsilon.$$

We have that $L(\gamma_{i_k}) < S + \varepsilon$ for sufficiently large k. Hence,

$$\sum_{j=1}^{m} d(\gamma(t_{j-1}), \gamma(t_j)) < S + 2\varepsilon.$$

As $\varepsilon > 0$ was arbitrary, it follows that

$$\sum_{j=1}^{m} d(\gamma(t_{j-1}), \gamma(t_j)) \leq S.$$

Taking the supremum over all partitions completes the proof. $\square$

In other words, L is a lower semi-continuous funtional on the collection of all paths in X with respect to pointwise convergence.

Definition 3.3. We call a metric d intrinsic if for every $x, y \in X$ we have

$$d(x, y) = \inf\{L(\gamma) \mid \gamma \text{ is rectifiable with } \gamma(a) = x \text{ and } \gamma(b) = y\}.$$

The shortest path between x and y is any curve γ that realises this infimum. If every pair of points in X can be joined by such a path we call our metric d strictly intrinsic.

When d is intrinsic, we say that (X, d) is a length space.

Example 3.4. Consider the punctured plane $\mathbb{R}^2 \setminus \{0\}$ with the Euclidean metric. This is readily seen to be a length space. But in this space the metric is not strictly intrinsic because for points opposite each other with respect to the origin, we cannot obtain the infimum.

This example highlights an important issue, when can we ensure the existence of shortest paths between two points in a length space.

Proposition 3.5. *Suppose (X, d) is a compact metric space and let $x, y \in X$ be points that can be connected by at least one rectifiable curve. Then there exists a shortest path between x and y.*

Proof. (Burago, Burago, Ivanov et al. 2001, p. 49) Let L_{inf} denote the infimum of lengths of rectifiable paths joining x and y. Therefore we can find a sequence $\{\gamma_i\}$ of rectifiable paths joining x and y such that

$$L(\gamma_i) \to L_{\text{inf}}$$

as $i \to \infty$. By reparameterizing each γ_i to have constant speed on the interval $[a, b]$, the sequence becomes a family of equicontinuous mappings. Since (X, d) is a compact metric space, it follows by Arzelà–Ascoli ((Burago et al. 2001, p. 47)) that we can find a subsequence $\{\gamma_{i_j}\}$ that converges uniformly to a path γ connecting x and y. By Proposition 3.2 it then follows that

$$L(\gamma) \leq \lim_{j \to \infty} L(\gamma_{i_j}) = L_{\text{inf}}.$$

Thus $L(\gamma) = L_{\text{inf}}$. □

Definition 3.6. Let X be a metric space. We say X is locally compact if for every $p \in X$, there exists an open U containing p such that $\overline{U}$ is compact.

From the previous proposition we immediately get the following local result.

Theorem 3.7. *Let (X, d) be a locally compact length space. For any point $p \in X$ we can find a neighbourhood V of p such that for any two points $x, y \in V$ there is a shortest curve joining these points.*

Proof. By local compactness, there exists an open neighbourhood U of p such that $\overline{U}$ is compact. Since U is open, we can find $r > 0$ such that $B(p, r) \subseteq U$. We define V as a smaller ball $B(p, \frac{r}{3})$. Thus, for any $x, y \in V$, $d(x,y) < \frac{2r}{3}$. Any path with length realising the infimum must then remain within the compact set $\overline{B(p,r)}$ since any path exiting would have a length of at least $\frac{4r}{3}$. The result then follows from Proposition 3.5. $\square$

We can establish a global version of this theorem if we additionally ask for completeness. Our previous example of the punctured plane shows why this condition is necessary.

Theorem 3.8. *Let (X, d) be a complete, locally compact length space. Then d is strictly intrinsic.*

Proof. A proof can be found in (Burago et al. 2001, p. 50) $\square$

4 Alexandrov

4.1 Triangulation

In classical differential geometry, curvature can be defined using the second derivatives of the metric. This approach fails if we wish to work with surfaces with corners like those found in a polyhedral surface. In the mid-20th century a great deal of work was done by A. D. Alexandrov and the wider school of Leningrad to develop a rich theory for singular surfaces via geometric methods. A systematic presentation of this program can be found in (Aleksandrov & Zalgaller 1967), which is the main reference for this chapter.

Let (R, g) be a Riemannian surface. The distance function induced by g gives us a length space and R inherits local compactness from its manifold structure. Thus Theorem 3.7 guarantees that every point has a neighbourhood within which any two points can be joined by a shortest path. By restricting U to a sufficiently small geodesic ball where the exponential map is a diffeomorphism, we ensure that our shortest path is also guaranteed to be unique. Let us also assume that

$$|\omega(U)| = \int_U |K| \, d\mu$$

is finite.

Consider a geodesic triangle T in a Riemannian surface with interior angles α, β, γ. The Gauss-Bonnet formula states

$$\iint_T K \, dS + \sum_{i=0}^{3} \int_{s_i}^{s_{i+1}} k_g \, ds + \sum_{i=0}^{3} \theta_i = 2\pi\chi(T)$$

where θ_i are the external angles at the vertices. We can immediately simplify this by noting a few points. We are bounding a domain homeomorphic to a closed disc, the boundary of our triangle is formed by geodesics and the sum of external and interior angles is π. This yields

$$\iint_T K\,dS + (\pi - \alpha) + (\pi - \beta) + (\pi - \gamma) = 2\pi.$$

Hence we can write the total curvature of T as follows

$$\omega(T) = \iint_T K\,dS = \alpha + \beta + \gamma - \pi = \delta(T). \tag{2}$$

Where we call the quantity $\delta(T)$ the excess of the triangle T. We see that on a flat Euclidean surface this is clearly just zero, but suppose our triangle was on a surface with positive curvature such as a sphere. Then we would have $\delta(T) > 0$. Establishing a link between the curvature K to just the angles of our triangle. This leads us to the idea that we may be able to extend our concept of curvature to more general surfaces if we start with the concept of angle.

Let (R, d) be an arbitrary length space and suppose L and M are two curves in R with common origin O. From our curves take two distinct points $X \in L$ and $Y \in M$. If we construct the triangle OXY on the plane, denoting the distance between two points by $|OX|$. Then the angle at O is simply

$$\nu(X, Y) = \arccos\left(\frac{|OX|^2 + |OY|^2 - |XY|^2}{2\,|OX|\,|OY|}\right).$$

In this spirit we define the upper angle at O between L and M by

$$\overline{\alpha} = \limsup_{X,Y\to O} \nu(X, Y) = \limsup_{X,Y\to O} \arccos\left(\frac{d(O,X)^2 + d(O,Y)^2 - d(X,Y)^2}{2d(O,X)d(O,Y)}\right).$$

Taking $\liminf$ instead defines the lower angle $\underline{\alpha}$ at O. If values coincide we say the angle at O between the two curves exists as we denote it $\alpha = \overline{\alpha} = \underline{\alpha}$. We define the upper excess of a triangle by

$$\overline{\delta}(T) = \overline{\alpha} + \overline{\beta} + \overline{\gamma} - \pi$$

We choose to work with the upper excess because the angle may not exist. However the upper limit necessarily does with $0 \leq \underline{\alpha} \leq \overline{\alpha} \leq \pi$.

Let us briefly return to the Riemannian case. Suppose we took a finite collection of geodesic triangles T_i in our domain U that did not share any interior points. Then from (2) we see that

$$\sum_{i=1}^{m} \delta(T_i) = \sum_{i=1}^{m} \omega(T_i) \leq \sum_{i=1}^{m} \omega^+(T_i) \leq \omega^+(U) = C < \infty.$$

Where we can get arbitrarily close to C by taking a finer covering of sufficiently small triangles. This remark directly suggests how we are going to define our more general surface.

Definition 4.1. A triangle in the space R consists of three distinct points and three shortest curves joining these points.

To avoid any initial complications we first restrict ourselves to a specific type of triangle.

Definition 4.2. Let $T \subset R$ be a triangle whose sides form a simple curve in R and bound a region G homeomorphic to a disc. Additionally, suppose that no two points on the boundary of T can be joined by a curve, lying outside G at all points, which is shorter than the shorter section of the boundary joining the points. Then T is called a simple triangle in R.

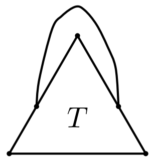

Figure 3: Example of a simple triangle.

Definition 4.3. A space (R, d) is a surface of bounded integral curvature if it satisfies the following three conditions:

1. (R, d) is a length space.
2. Each point in R has a neighbourhood G homeomorphic to the open disc.
3. There exists a constant $C(G) < \infty$ such that for any finite system $\{T_1, \ldots, T_n\}$ of pairwise nonoverlapping simple triangles contained in G we have that

$$\sum_{i=1}^{n} \overline{\delta}(T_i) \leq C(G).$$

The first two conditions ensure that (R, d) is a two-dimensional manifold without boundary. Condition 1 gives us connectedness and condition 2 gives us local compactness. Thus, since we also have a metric already, R must have a countable basis. The third requirement is given in the weakest form possible and can be called the condition of boundedness of the curvature. Chapter 4 of (Aleksandrov & Zalgaller 1967) gives bounds that are true in a more general sense. For example, a wider class of triangles.

We can now define a curvature measure ω. The positive part of the curvature, $\omega^+(U)$, of an open set U is defined as

$$\omega^+(U) = \sup \left(\sum_{i=1}^{n} \delta^+(T_i) \right).$$

Where the supremum is over all finite systems of non-overlapping simple triangles in U

and $\delta^+(T_i) = \max(0, \delta(T_i))$ is the positive excess. The negative part of the curvature is defined in the same way with $\delta^-(T_i) = \max = (0, -\delta(T_i))$. To extend this an arbitrary Borel set B we put by definition

$$\omega^+(B) = \inf_{B \subset U} \omega^+(U)$$

and again define $\omega^-(B)$ in the same way. We then set $\omega = \omega^+ - \omega^-$. We remark that in the case of a Riemannian surface (M, g), we simply have that $\omega = K d\mu$.

Unlike the smooth case, we do not always have $\omega(T) = \alpha + \beta + \gamma - \pi$ for some geodesic triangle T.

A natural question to ask is if our notion of curvature is even well defined. We have defined ω by using δ, which may cause issues if angles cannot be defined at every point. However, it is the case that if (R, d) is a surface of bounded integral curvature then between any two shortest paths starting from one point the angle exists. This fact is not obvious from the current definition.

4.2 Approximation

One of the main goals of this new definition was to develop a system of concepts and methods that could handle equally the investigation of Riemannian surfaces, polyhedra and general convex surfaces. Our original definition of a surface of bounded integral curvature was an axiomatic method, by making as few choices of properties and building up our theory. This does not lead to a definition that is practically easy to work with.

A key insight by the mathematicians in Leningrad was that this space could be achieved through an approximation method by polyhedral metrics. Since any polyhedral surface is the uniform limit of a sequence of Riemannian surfaces one can view the space of surfaces with bounded integral curvature as closing the class of Riemannian surfaces by adding all metrized surfaces whose metric in a neighbourhood of each point can be uniformly approximated by Riemannian metrics such that integrals of the absolute value of the Gauss curvature is uniformly bounded.

A space with a polyhedral metric is one in which each point has a neighbourhood isometric to the lateral surface of a cone where θ is the complete angle at the apex. For points in the interior of a face or edge, the neighbourhood just unfolds into a flat disc (alternatively angle $\theta = 2\pi$). At vertices we have a difference $2\pi - \theta$, we take the curvature at this point to be this difference. We define the curvature ω of any set is defined as the sum of the curvatures of the vertices belonging to that set.

Definition 4.4. Suppose (ρ_n) is a sequence of metrics in an open set U of a two-dimensional manifold R. If uniformly for all points $x, y \in U$

$$\lim_{n\to\infty} \rho_n(x, y) = \rho(x, y)$$

and if the limiting function ρ is also a metric, then we say that the metrics ρ_n converge to the metric ρ.

For the remainder of the chapter we suppose (R, ρ) is a two-dimensional manifold such that topology induced by ρ is compatible with the manifold topology.

Theorem 4.5. *Let (R, ρ) be a length space where each point in R has a neighbourhood homeomorphic to the disc. Assume that for any point $x \in R$ we can find a neighbourhood U and a sequence of polyhedral metrics ρ_n in U that converges to ρ and is such that the sequence $\omega^+_{\rho_n}(U)$ is bounded. Then (R, ρ) is a surface with bounded integral curvature.*

The original proof can be found in (Aleksandrov & Zalgaller 1967, Chapter 4), here we present a simpler proof found in (Reshetnyak 1993, p. 94).

Definition 4.6. A two-dimensional manifold Q with intrinsic metric homeomorphic to a closed disc is called a convex cone if Q is isometric to a convex domain on the cone $Q(\theta)$ where $\theta < 2\pi$.

Let $U \subset R$ be an open domain. Assume that we have a sequence of metrics (ρ_n) in U such that the following are satisfied.

1. The metrics ρ_n converge to the metric ρ.
2. The space (U, ρ_n) is a polyhedron.
3. There exists a constant $A < \infty$ such that $\omega^+_{\rho_n}(U) < A$ for all metrics in our sequence.
4. The set functions $\omega^+_{\rho_n}(E)$ converge weakly to a set function $\omega_0(E)$.

We will require the following lemma which we state without proof, one can be found in (Reshetnyak 1962).

Lemma 4.7. *Suppose the above conditions are satisfied. Let Γ be a simple closed curve in U, and $G \subset U$ be the open set whose boundary is Γ. Further assume that $\omega_0(G) < 2\pi$ and that Γ is rectifiable with respect to ρ. Then there is a convex cone Q such that $\omega(Q) \leq \omega_0(G)$, and a contraction ϕ of Q onto the domain $G \cup \Gamma$ which maps the boundary of Q one to one and preserves arc lengths.*

Proof. In order to prove Theorem 4.5 we begin by assuming a two-dimensional manifold R with intrinsic metric ρ meets the conditions of Lemma 4.7. Take an arbitrary point $x \in R$ and let U be a neighbourhood of x homeomorphic to a disc. Take ρ_n to be a sequence of polyhedral metrics in U that converge to ρ such that $\omega^+_{\rho_n}$ is bounded. By passing to a suitable subsequence if necessary we can also assume $\omega_{\rho_n}^{+}(E)$ converge weakly in U to $\omega_0(E)$.

Let T be a simple triangle in U, denote its vertices by A, B, C where $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$ are the respective upper angles. The sides of T form a simple closed curve Γ and we call the interior of the triangle G. We wish to prove that

$$\overline{\delta}(T) = \overline{\alpha} + \overline{\beta} + \overline{\gamma} - \pi \leq \omega_0(G).$$

In this case we could take any finite collection of pairwise non-overlapping simple triangles $\{T_1, \ldots, T_m\}$ in U homeomorphic to a disc. We would have for each i that

$$\overline{\delta}(T_i) \leq \omega_0(G_i).$$

Hence

$$\sum_{i=1}^{m} \overline{\delta}(T_i) \leq \sum_{i=1}^{m} \omega_0(G_i) = \omega\left(\bigcup_{i=1}^{m} G_i\right) < \infty.$$

Establishing that the two-dimensional manifolds considered were indeed surfaces of bounded integral curvature.

Let us then prove the inequality. Since we have $\overline{\alpha}, \overline{\beta}, \overline{\gamma} \leq \pi$ we can suppose that $\omega_0(G) < 2\pi$. Lemma 4.7 tells us there is convex cone Q such $\omega(Q) \leq \omega_0(G)$. Denote A_0, B_0, C_0 to be the points on the boundary of Q such that $\phi(A_0) = A, \phi(B_0) = B, \phi(C_0) = C$.

Since ϕ preserves arc lengths on the boundary we have that $L([A_0B_0]) = L([AB]) = \rho(A, B)$. Now, as we know ϕ is a contraction we have that $\rho(A, B) \leq \rho(A_0, B_0)$.

Therefore, $L([A_0B_0]) \leq \rho(A_0, B_0)$. Since the length of a curve cannot be shorter than the shortest distance, we must have equality. Showing $[A_0B_0]$ is a shortest curve. An identical argument shows the same for $[B_0C_0]$ and $[B_0C_0]$. With this in mind we can let $\alpha_0, \beta_0, \gamma_0$ denote the bounding angle at the points A_0, B_0, C_0. It follows by Gauss Bonnet and the curvature bound on our convex cone that

$$\alpha_0 + \beta_0 + \gamma_0 - \pi = \omega(Q) \leq \omega_0(G).$$

Let $X \in [AB]$ and $Y \in [AC]$ be arbitrary points on our triangle. Let $\tilde{X}$ and $\tilde{Y}$ be the corresponding points so that $\phi(\tilde{X}) = X$ and $\phi(\tilde{Y}) = Y$. Define $z_0 = \rho(\tilde{X}, \tilde{Y}), z = \rho(X, Y)$. Since ϕ is a contraction we have $z \leq z_0$. So if on the plane we construct triangles with sides x, y, z and x, y, z_0 we will have $\nu(X, Y) \leq \nu_0(X, Y)$. Hence

$$\overline{\alpha} = \limsup_{X,Y \to A} \nu(X, Y) \leq \alpha_0.$$

In a similar fashion we can prove $\overline{\beta} \leq \beta_0$ and $\overline{\gamma} \leq \gamma_0$. The theorem follows since we now have

$$\overline{\delta}(T) = \overline{\alpha} + \overline{\beta} + \overline{\gamma} - \pi \leq \alpha_0 + \beta_0 + \gamma_0 - \pi = \omega(Q) \leq \omega_0(G)$$

as required. □

We also have a converse to this theorem.

Theorem 4.8. *Let (R, ρ) be a surface with bounded integral curvature. For any point x of it we can construct a neighbourhood U homeomorphic to a closed disc and a sequence of metrics (ρ_n) defined in U such that (U, ρ_n) is a polyhedron homeomorphic to a disc. The sequence of metrics converges to ρ and there is a constant $C < \infty$ such that for each n we have*

$$|\omega_{\rho_n}|(U) + |\kappa_{\rho_n}|(\partial U) \leq C.$$

The proof is extensive and is contained in (Aleksandrov & Zalgaller 1967, Chapter 3). We make a few remarks. The symbol $|\kappa_\rho|(\partial U)$ denotes the absolute turn of the boundary of the polyhedron (U, ρ). As we will see below, we will only consider the case when it is a Riemannian manifold. In this case it is simply evaluated as the integral of the modulus of the geodesic curvature of the boundary with resepct to curve length.

Theorem 4.8 gives a necessary condition which is also sufficient. However in the sufficient case one may prefer to work with Theorem 4.5 which weakens the conditions on ρ_n.

Starting from the approximation of the metric by polyhedral metrics simplifies many arguments, see subsection 6 in (Aleksandrov & Zalgaller 1967, Chapter 4) for an example of how this is applied to show the existence of an angle between arbitrary shortest arcs. Something we could not assume true beforehand. Since any polyhedral surface is the uniform limit of a sequence of Riemannian surface, the previous two theorems remain true if we consider Riemannian metrics instead.

5 Reshetynak

Let (M, g) be a Riemannian surface. In chapter 2 we saw that it is always possible to write g as

$$g = \lambda(x, y)|dz|^2$$

with respect to a local complex coordinate $z = x + iy$ on an open set U. We have also already shown that in these coordinates we have

$$K = -\frac{1}{2\lambda}\Delta \log \lambda.$$

Solving by means of the standard formula for the solution to the Poisson equation, see (Gilbarg & Trudinger 1998, p. 55), gives us the formula

$$\log \lambda(z) = \frac{1}{\pi} \int_U \log \frac{1}{|z-\xi|} K(\xi)\lambda(\xi)\, d\xi + h(z),$$

where $h(z)$ is a harmonic function. In chapter 4 we remarked that the curvature measure for a Riemannian surface is $\omega = K d\mu$. This gives

$$\log \lambda(z) = \frac{1}{\pi} \int_U \log \frac{1}{|z-\xi|}\, d\omega(\xi) + h(z).$$

This leads to one of the principle ideas of Reshetynak's work, that on a surface of bounded integral curvature we can also have Riemannian metrics of this form, but where ω is a signed measure. It will turn out to be the curvature measure we defined in Chapter 4.1.

5.1 Bounded Rotation

Let us begin by giving a brief overview of a subclass of the class of rectifiable curves of the plane, curves of bounded rotation.

Definition 5.1. Let P be a piecewise linear curve made up of segments $\{a_i\}$ and vertices $\{A_i\}$. The absolute value $\psi_i \in [0, \pi]$ of the angle between segments a_i and a_{i+1} is called the absolute rotation at A_i. We define the sum

$$|\kappa|(P) = \sum_{i=1}^{m-1} \psi_i$$

as the absolute rotation of the curve P

Definition 5.2. Let $K = \{\gamma(t) : a \le t \le b\}$ be a curve in the plane and P a piecewise linear curve, where $A_0, \dots, A_m$ are the ordered vertices. We say P is inscribed in K if we can find a sequence of parameters $t_0 < \cdots < t_m$ such that $A_j = \gamma(t_j)$ for all $j \in \{0, \dots, m\}$.

Definition 5.3. The absolute rotation $|\kappa|(K)$ of a curve K is the supremum of the absolute rotation of the broken lines inscribed in K. We say it is of bounded rotation if $|\kappa|(K)$ is finite.

Lemma 5.4. *Let (K_n) be a sequence of curves converging to K.*
Then $|\kappa|(K) \le \liminf |\kappa|(K_n)$.

Proof. (Alexandrov & Reshetnyak 1989, p. 121) Let P be a piecewise linear curve inscribed in K, and let P_n be piecewise linear curves inscribed in K_n such that the associated sequences of vertices converge to the vertices of P. Clearly, $|\kappa|(P_n) \to |\kappa|(P)$ and $|\kappa|(P_n) \le |\kappa|(K_n)$, so $|\kappa|(P) \le \liminf |\kappa|(K_n)$, and as P is arbitrary, $|\kappa|(K) = \sup |\kappa|(P) \le \liminf |\kappa|(K_n)$. □

Lemma 5.5. *Let (P_n) be a sequence of piecewise linear curves inscribed in K that converges to K. Then the $L(P_n)$ converges to $L(K)$ and $|\kappa|(P_n)$ converges to $|\kappa|(K)$.*

Proof. Since for every m, we have by definition that $|\kappa|(K) \geq |\kappa|(P_n)$ it follows

$$|\kappa|(K) \geq \limsup |\kappa|(P_n)$$

Using Lemma 5.4 provides the upper bound and thus the limit exists as required. See (Alexandrov & Reshetnyak 1989, p. 30) for a proof of the lengths. □

One interesting remark about this lemma is that since the absolute rotation of a closed convex piecewise linear curve is 2π, the absolute rotation of a circle is 2π as we would expect. To round out this section we aim to prove that all curves of bounded rotation are rectifiable. We will require the following inequality due to Alexandrov.

Theorem 5.6. *Let K be a curve with endpoints z_1, z_2 such that $|\kappa|(K) < \pi$. Then*

$$L(K) \leq \frac{|z_1 - z_2|}{\cos\left(\frac{|\kappa|(K)}{2}\right)}.$$

Proof. We adapt the proof found in (Alexandrov & Reshetnyak 1989, p.151). Let us first make the following observation, if A is an interior vertex of a piecewise linear curve P and P_1, P_2 are obtained from P by splitting at A. Then if α is the angle at A, we have

$$|\kappa|(P) = |\kappa|(P_1) + |\kappa|(P_2) + \alpha.$$

Hence, for any decomposition $a = t_0 < \cdots < t_n$ of a parametrization of curve K it follows that $t \mapsto |\kappa|(K|_t)$ is non-decreasing since

$$\sum_i |\kappa|(K|_{t_i, t_{i+1}}) \leq |\kappa|(K). \tag{3}$$

From *Lemma*5.5 we know it is enough to prove the formula for piecewise linear curves P. Define $\gamma : [0, L(P)] \to \mathbb{C}$ be a curve length parametrization of a piecewise linear curve, giving us that its left derivative has unit norm. To simplify, let us apply a rotation, suppose the first segment of P is directed by $(1,0)$. Let e be a unit vector with principal argument $\frac{|\kappa|(P)}{2}$, this choice is made so we are centered in the range of directions our curve can take. From our observation before that $|\kappa|(K|_t)$ is non-decreasing we have for any $t \in [0, L(P)]$ the angle between the left derivative $z'_l(t)$ and e is less than $\frac{|\kappa|(P)}{2}$. In other words, $\langle z'_l(t), e\rangle \geq \cos\left(\frac{|\kappa|(K)}{2}\right)$. We conclude by seeing that

$$|z_1 - z_2| \geq \langle z_1 - z_2, e\rangle = \left\langle \int_0^{L(P)} z'_l(t)dt, e \right\rangle = \int_0^{L(p)} \langle z'_l, e\rangle \, dt \geq L(p)\cos\left(\frac{|\kappa|(p)}{2}\right).$$

□

Theorem 5.6 gives us an upper bound on the length of a curve provided its absolute rotation is not too big. We would like to break up a general curve and then apply this inequality to each individual segment. Consider a right neighbourhood of a point $t_0 \in [a, b), [t_0, t_0 + \delta]$. Let $t_n \to t_0, t_0 < t_n < b$. By the observation (3) made in the proof of Theorem 5.6 we have

$$|\kappa|(K|_{t_0,b}) - |\kappa|(K|_{t_n,b}) \geq |\kappa|(K|_{t_0,t_n}) \geq 0.$$

Lemma 5.4 and that $|\kappa|(K|_t)$ is non-decreasing now yields that $|\kappa|(K|_{t_n,b}) \to |\kappa|(K|_{t_0,b})$ and thus

$$\lim_{n\to\infty} |\kappa|(K|_{t_0,t_n}) = 0. \tag{4}$$

Lemma 5.7. *Let K be a curve parametrized on the interval $[a, b]$ such that $|\kappa|(K) = A < \infty$. Then for all $\varepsilon > 0$, there exists $N \leq \frac{A}{\varepsilon} + 1$ and a decomposition $a = t_0 < \cdots < t_N = b$ such that for every i, $|\kappa|(K|_{t_i,t_{i+1}}) \leq \varepsilon$.*

Here the important thing is that whilst the decomposition itself depends on the curve, N only depends on A and ε.

Proof. Let $\varepsilon > 0$. We define t_i iteratively. Set $t_0 = a$, to find subsequent points t_k, look for the smallest $t \in (t_{k-1}, b]$ such that

$$|\kappa|(K|_{t_{k-1},t}) = \varepsilon.$$

This is possible by (4). Set the final point as $t_N = b$, by construction the first $N - 1$ segments satisfy $|\kappa|(K_{t_{k-1},t_k}) = \varepsilon$, the final segment $|\kappa|(K_{t_{N-1},t_N}) \leq \varepsilon$. It follows that

$$A \geq |\kappa|(K) \geq \sum_{k=1}^{N} |\kappa|(K|_{t_{k-1},t_k}) \geq \sum_{k=1}^{N-1} |\kappa|(K|_{t_{k-1},t_k}) = (N-1)\epsilon.$$

Hence, $N \leq \frac{A}{\varepsilon} + 1$ as required. □

Theorem 5.8. *Any curve of bounded rotation is rectifiable.*

Proof. By Lemma 5.7 we have that a curve of bounded rotation can be decomposed into a finite number of shorter curves with absolute rotation less than π. Applying Theorem 5.6 to each individually then gives the result immediately by concatenation. □

We conclude with a theorem that we will need later which we state without proof, technical details can be found in (Alexandrov & Reshetnyak 1989, p. 146).

Theorem 5.9. *Take any sequence of curves contained in a bounded domain with absolute rotation bounded uniformly from above. Then there exists a converging subsequence with converging length.*

5.2 Subharmonic Functions

Definition 5.10. Let $E \subset \mathbb{R}^n \cup \{\infty\}$. A function $f : E \to [-\infty, \infty]$ is called upper semicontinuous on E if $\{x \in E : f(x) < a\}$ is a relatively open subset of E for all $a \in \mathbb{R}$.

Example 5.11. The floor function $f(x) = \lfloor x \rfloor$, is everywhere upper semicontinuous. But clearly not everywhere continuous.

Definition 5.12. A function $u : \Omega \to [-\infty, \infty)$ is called subharmonic on a domain $\Omega \subset \mathbb{R}^m$ if the following are true:

1. u is upper semicontinuous on Ω.
2. $u \not\equiv -\infty$ on any component of Ω.
3. For any point $z \in \Omega$ there exists sufficiently small radius r_0 such that for all $0 < r < r_0$ and $\overline{B(z,r)}$ we have

$$u(z) \leq \frac{1}{c_m r^{m-1}} \int_{S(z,r)} u(x)\ d\sigma(x).$$

   Where c_m is the surface area of the unit $(m-1)$ sphere in $\mathbb{R}^m$.

We now state Riesz's Theorem (Armitage & Gardiner 2012, p. 105) in special case for $\mathbb{R}^2$.

Theorem 5.13. *Suppose $u(x)$ is subharmonic and not identically $-\infty$ in a domain $\Omega \subset \mathbb{R}^2$. Then there exists a unique Borel measure μ in Ω such that for any compact subset $K \subset \Omega$ we have*

$$u(x) = \int_K \log|x - \xi|\ d\mu(\xi) + h(x).$$

Let ω be a signed measure with support contained in an open disc and defined on the Borel σ-algebra of that disc. Let M be a bounded domain in the plane, h a harmonic function defined over M. We set

$$\lambda(z) = \lambda(z; \omega, h) = \exp\left(\frac{1}{\pi}\int_M \log\left|\frac{1}{z-\xi}\right|\ d\omega(\xi) + h(z)\right).$$

Since ω is a signed measure, we have $\omega = \omega^+ - \omega^-$ by the Jordan Decomposition Theorem.

The logarithm of the function $\lambda(z;\omega,h)$ is equal to the difference of two functions

$$u_1(z) = \frac{1}{\pi}\int_M \ln|z-\xi|\,d\omega^-(\xi) + h(z),$$
$$u_2(z) = \frac{1}{\pi}\int_M \ln|z-\xi|\,d\omega^+(\xi).$$

By Riesz's Theorem we have that both of these functions are subharmonic. Hence, $\log\lambda$ is given by the difference of two subharmonic functions.

Given such a $\lambda(z)$ defined over a bounded domain M we would like to define a notion of distance in a similar fashion to Riemannian geometry. Consider the metric defined as

$$ds^2 = \lambda(x,y)(dx^2+dy^2) = \lambda(z)|dz|^2$$

For some rectifiable curve in M parametrised by curve-length we would like to define the length of γ with respect to ds^2 as

$$L_\lambda(\gamma) = \int_0^l \sqrt{\lambda(\gamma(s))}\,ds.$$

But by how we have defined $\lambda(z)$ it is only finite almost everywhere. Since a rectifiable curve is a set of measure zero, it is not clear immediately that $\lambda(\gamma(s))$ is finite for a set of values that is not a null set. To resolve this, we employ the following lemma which we state without proof. We refer the reader to (Fillastre 2023, p. 87).

Lemma 5.14. *Let $u(z)$ be a subharmonic function in a domain G. Then $-\infty \leqslant u(z) < \infty$. Let E be the set of those z for which $u(z) = -\infty$. Then for any $\varepsilon > 0$ and any $\alpha > 0$ there is a sequence of open discs $(B_m = B(c_m, r_m))$ such that*

$$E \subset \bigcup_{m=1}^{\infty} B_m$$

and

$$\sum_{m=1}^{\infty} r_m^\alpha < \varepsilon.$$

We have that $\lambda(z)$ is problematic for points where $u_1(z) = -\infty$ or both u_1 and u_2 take the value $-\infty$. But Lemma 5.14 tells us that given $\varepsilon > 0$, we can cover these points by a sequence of discs for which the sum of the radii is less than ε. It then follows from Proposition 4.70 and Lemma 4.72 in (Fillastre 2023, p. 88) that the set of parameters $s \in [0,l]$ for which the curve lies in these discs has Lebesgue measure 0. Hence $\lambda(\gamma(s))$ is defined and finite for almost all $s \in [0,l]$. It is also clear that $\lambda(\gamma(s))$ is non-negative and measurable by virtue of it being the difference of two subharmonic functions. Hence the integral makes sense and $L_\lambda(\gamma)$ is well defined. We now define the distance as

$$\rho_\lambda(z_1, z_2) = \inf L_\lambda(\gamma),$$

where the infimum is taken over the set of all curves of bounded rotation.

We call such a distance defined by a function $\lambda(z) = \lambda(z; \omega, h)$ subharmonic. We call the measure ω that determined this distance the curvature.

It may be the case that ρ_λ takes the value ∞ for some points, in particular, we have the following computation.

$$\frac{1}{\pi}\int_M \log\left|\frac{1}{z-\xi}\right| \, d\omega(\xi) = \frac{\omega(\{z_0\})}{\pi}\log\left|\frac{1}{z-z_0}\right| + \frac{1}{\pi}\int_{M\setminus\{z_0\}} \log\left|\frac{1}{z-\xi}\right| \, d\omega(\xi).$$

From which we see that $\sqrt{\lambda(z)}$ can be represented as

$$\sqrt{\lambda(z)} = |z-z_0|^{-\frac{\omega(\{z_0\})}{2\pi}}\sqrt{\lambda_0(z)}.$$

If $\omega(\{z_0\}) > 2\pi$ then the integral of $\sqrt(\lambda(z))$ along any curve passing through z_0 yields ∞. However, if $\omega(\{z_0\}) = 2\pi$ then the convergence depends on the behaviour of $\lambda_0(z)$ in the limit $z \to z_0$.

With this in mind, let us if $\rho_\lambda(z_0, z) = \infty$ then z_0 is a point at infinity. Above we remarked that if $\omega(\{z_0\}) = 2\pi$ then it was not possible to determine a priori if it was a point at infinity or not. We have the following example that shows this.

Example 5.15. Define ω to be $\omega(E) = 2\pi$, if $0 \in E$, and $\omega(E) = 0$ when $0 \notin E$. Take

$$\lambda(z;\omega) = \left|\frac{1}{z}\right|^2.$$

The length of the circle $\{|z| = r\}$ is 2π for all r. We have that the origin is a point at infinity in the plane. In some sense what we have done is punch a hole in the origin and pull the edges of that hole forever, turning it into a long tube. More formally, the plane endowed with ρ_λ is isometric to the surface of $S^1 \times \mathbb{R}$.

5.3 Distances Convergence Theorem

Definition 5.16. A sequence of measures ω_n in a topological space X converges weakly to a measure ω_0 if for any continuous bounded function f on X we have

$$\int_X f(x)\, d\omega_n(x) \xrightarrow[n\to\infty]{} \int_X f(x)\, d\omega_0(x).$$

We denote this by $\omega_n \rightharpoonup \omega_0$.

We state the main goal of this subchapter, the Distances Convergence Theorem.

Theorem 5.17. *Let M be a bounded closed domain in the plane. Suppose the boundary consists of a finite number of simple closed curves of bounded rotation that are pairwise non-intersecting. Let ω_n^1, ω_n^2 be two sequences of positive measures with support in a given disc $\{|z| \leq R\}$. Suppose that as $n \to \infty$, $\omega_n^1 \rightharpoonup \omega_0^1$ and $\omega_n^2 \rightharpoonup \omega_0^2$. We define*

$$\omega_n = \omega_n^1 - \omega_n^2,\ \omega_0 = \omega_0^1 - \omega_0^2.$$

Let $\lambda_n(z) = \lambda(z; \omega_n)$ be a sequence defined by these measures and let ρ_{λ_n} be the distance defined over M such that $ds^2 = \lambda_n(z)(dx^2 + dy^2)$. Suppose that

$$\omega_0^1(\{z\}) < 2\pi\ , \forall z \in M.$$

Then (ρ_{λ_n}) converges uniformly to ρ_{λ_0}.

We begin by proving two lemmas which will help simplify our approach.

Lemma 5.18. *Let ω_n be positive measures over the domain $D = \{|z| \leq R\}$ such that $\omega_n \rightharpoonup \omega_0$. For each fixed n, let $(\omega_{m,n})$ be a sequence of positive measures where $\omega_{m,n} \rightharpoonup \omega_n$. There exists a sequence of indices (i_n) such that when $m_n > i_n$ we have that $\omega_{m_n,n} \rightharpoonup \omega_0$.*

Proof. (Reshetnyak 1960*a*, p. 97)
Since (ω_n) converges weakly we define $A = \sup_n\{\omega_n(D)\} < \infty$. By picking a constant test function in Definition 5.16 we also have

$$\omega_{m,n}(D) \xrightarrow[m\to\infty]{} \omega_n(D).$$

Take $\tilde{i}_n$ so that when $m > \tilde{i}_n$ we have $\omega_{m,n}(D) < A + 1$. To simplify the proof we will use the Weierstrass Approximation Theorem; any continuous function can be approximated uniformly by polynomials with rational coefficients. Let $\mathbb{P} = \{P_1, P_2, \dots, P_k, \dots\}$ be the set of all polynomials with rational coefficients over the plane. For any k it is clear that

$$\int_D P_k\, d\omega_{m,n} \xrightarrow[m\to\infty]{} \int_D P_k\, d\omega_n.$$

Now take $i_n \geq \tilde{i}_n$ so that when $m \geq i_n$ we have for $k = 1, \dots, n$ that

$$\left| \int_D P_k\, d\omega_{m,n} - \int_D P_k\, d\omega_n \right| < \frac{1}{n}.$$

Consider now any arbitrary polynomial P_r in $\mathbb{P}$, by assumption we have that

$$\int_D P_r\, d\omega_n \xrightarrow[n\to\infty]{} \int_D P_r\, d\omega_0.$$

Let $m_n \geq i_n$ for any n. When $n \geq r$ we now have

$$\left| \int_D P_r \, d\omega_{m_n,n} - \int_D P_r \, d\omega_n \right| < \frac{1}{n};$$

from which the triangle inequality yields

$$\int_D P_r \, d\omega_{m_n,n} \xrightarrow[n\to\infty]{} \int_D P_r \, d\omega_0.$$

We have shown that $\omega_{m_n,n} \rightharpoonup \omega_0$ on $\mathbb{P}$. By construction, since $m_n > \tilde{i}_n$, we have that the sequence $(\omega_{m_n,n}(D))$ is bounded. The aforementioned Weierstrass Approximation Theorem tells us that $\omega_{m_n,n} \rightharpoonup \omega_0$. □

Lemma 5.19. *It is sufficient to prove the Distances Convergence Theorem just for the case when the measures ω_n^1, ω_n^2 have C^1 Lebesgue density.*

Proof. (Reshetnyak 1960*a*, p. 98) For each n, let $(\omega_{m,n}^1), (\omega_{m,n}^2)$ be sequences of positive measures with C^1 Lebesgue density such that

$$\omega_{m,n}^1 \rightharpoonup \omega_n^1, \; \omega_{m,n}^2 \rightharpoonup \omega_n^2.$$

Lemma 5.18 gives us a sequence (i_n) that ensures $\omega_{m_n,n}^j \rightharpoonup \omega_0^j$ $(j = 1, 2)$. Define

$$\omega_{m,n} = \omega_{m,n}^1 - \omega_{m,n}^2, \; \omega_0 = \omega_0^1 - \omega_0^2.$$

We have that $\omega_{m_n,n} \rightharpoonup \omega_0$. Let (x_n) and (y_n) be sequences such that $x_n \to x_0, y_n \to y_0$ where $\omega_0^1(\{x_0\}) < 2\pi$ and $\omega_0^1(\{y_0\}) < 2\pi$.
The measures $\omega_{m,n}$ give us distances $\rho_{\lambda_{m,n}}$ over M. Suppose now that the C^1 Lebesgue density case is proved. For sufficiently large n_0 we can ensure that

$$\omega_0^1(\{x_n\}) < 2\pi, \; \omega_0^1(\{y_n\}) < 2\pi.$$

So for all $n \geq n_0$ it follows that

$$\rho_{\lambda_{m,n}}(x_n, y_n) \xrightarrow[m\to\infty]{} \rho_{\lambda_n}(x_n, y_n). \tag{5}$$

Let us now specify a choice of $m_n > i_n$. Now not only do we get (5), but also

$$\rho_{\lambda_{m_n,n}}(x_n, y_n) \xrightarrow[m\to\infty]{} \rho_{\lambda_0}(x_0, y_0).$$

This follows since $\omega_{m_n,n}^j \rightharpoonup \omega_0^j$ and using fact they have C^1 Lebesgue density so we can apply the Distances Convergences Theorem in the restricted case. We now see that

$$|\rho_{\lambda_n}(x_n, y_n) - \rho_{\lambda_0}(x_0, y_0)| \leq \left|\rho_{\lambda_n}(x_n, y_n) - \rho_{\lambda_{m_n,n}}(x_n, y_n)\right| + \left|\rho_{\lambda_{m_n,n}}(x_n, y_n) - \rho_{\lambda_0}(x_0, y_0)\right|,$$

from which the result follows immediately. $\square$

In the interest of space we will only prove the Distances Convergence Theorem under a stronger restriction. Namely that for all $z \in M$ we have that

$$(\omega_0^1 + \omega_0^2)(\{z\}) < 2\pi.$$

This is known as the Intermediate Distances Convergence Theorem. The proof is built off of two key lemmas which we state without proof, both can be found in (Reshetnyak 1960*b*).

Lemma 5.20. (Lengths Convergence Lemma) *Under the hypothesis of the Distances Convergence Theorem, let $(\gamma_n), \gamma_0$ be curves of uniformly bounded rotation, such that $\gamma_n \to \gamma_0$ and*

$$\omega_0^1(\{z\}) < 2\pi, \ \forall z \text{ in } \gamma_0.$$

Then

$$L_{\lambda_n}(\gamma_n) \xrightarrow[n\to\infty]{} L_{\lambda_0}(\gamma_0), \text{ and } L_{\lambda_0}(\gamma_0) < \infty.$$

Lemma 5.21. *(Local Estimate of Shortest Arcs Rotations Lemma) Again under the hypothesis of the Distances Convergence Theorem, let the measures ω_n^1 and ω_n^2 have C^1 Lebesgue density. Then for any point $z \in M$ such that*

$$\omega_0^1(\{z\}) + \omega_0^2(\{z\}) < 2\pi,$$

there exists a constant A and a neighbourhood V such that the absolute rotation of any curve, shortest for ρ_{λ_n} and connecting two points of the closure of V, does not exceed A.

Corollary 5.22. *Let F be a closed subset of M such that*

$$\omega_0^1(\{z\}) + \omega_0^2(\{z\}) < 2\pi, \ \forall z \in F.$$

Then there exists $A < \infty$ such that for any shortest curve $\gamma \subset F$ for ρ_{λ_n}, we have $|\kappa|(\gamma) < A$.

Proof. (Reshetnyak 1960*a*, p. 100) By Lemma 5.21 every point $z \in F$ has a number $A(z)$ and a neighbourhood $V(z)$ around it such that the absolute rotation of any shortest curve for ρ_{λ_n}, connecting two points of $\overline{V(z)}$, does not exceed $A(z)$. We apply Heine-Borel to find a finite cover of F, denote these by

$$V_1 = V(p_1), \ldots, V_m = V(p_m).$$

Let A_0 be the largest of the numbers associated with these p_i and let $\gamma \subset F$ be a shortest curve for ρ_{λ_n}.

We now start by defining z_0 as the starting point of our curve γ. Clearly z_0 belongs to some neighbourhood V_j, let us say $z_0 \in V_{j_0}$. Now define z_1 as the furthest point of γ still belonging to $\overline{V_{j_0}}$. If z_1 is the final point of γ then we end the construction. Otherwise, z_1 must lie on the boundary of V_{j_0} and also belongs to some other neighbourhood V_{j_1}. Again define z_2 as the furthest along point and continue the construction. The sequence of neighbourhoods generated must all be different, if for instance $V_{j_l} = V_{j_k}$ and $l > k$, then $z_{j_{l+1}}$ will belong to $\overline{V_{j_k}}$ whilst being on γ. But it will lie further along that $z_{j_{k+1}}$ which contradicts the construction of this point.

Since our cover was finite, the construction ends in finite steps, so we have for some $p \leq m$, that z_p is the endpoint of γ. The absolute rotation of any segment $z_l z_{l+1}$ of γ is at most A_0, accounting for largest possible number of joints we could have it follows that the absolute rotation of γ itself can not be larger than

$$A = mA_0 + (m-1)\pi < \infty.$$

□

We are now in a position to prove the Intermediate Distances Convergence Theorem, we closely follow the detailed sketch found in (Fillastre 2023, p. 108).

Proof. By Lemma 5.19 we can assume that the measures ω_n^1, ω_n^2 have C^1 Lebesgue density. We will consider points x_n and x_0, where the x_n converge to x_0. Let us define γ_n as the curve that connects these two points for each n. We define σ_n in the same way for points y_n and y_0. Recall that the boundary of M consists of finitely many different curves of bounded rotation. We should be careful that we specify γ_n and σ_n such that in the limit, parts of these curves do not fall outside of M. One can imagine a scenario where this could occur, if locally the boundary thrusted sharply inwards.

To ensure we do not run into issues, we consider two cases. When x_0 is in the interior of M, we can simply take the straight segment $[x_0, x_n]$ for γ_n since for sufficient n we are entirely contained in a neighbourhood inside M. If instead, x_0 is a point on the boundary of M. We first find the point x_n' on the boundary closest to x_n. For a sufficiently large n we can ensure that x_n' belongs to the same boundary component of M as x_0. We form γ_n to be the straight segment $[x_n, x_n']$ together with the curve $x_n' x_0$ that goes along the boundary component of M (of course choosing the shortest out of the two options). We construct σ_n in an identical fashion. In both cases it is clear that γ_n and σ_n are curves of bounded rotation.

To prove the theorem we are going show that

$$\limsup \rho_{\lambda_n}(x_n, y_n) \leq \rho_{\lambda_0}(x_0, y_0) \leq \liminf \rho_{\lambda_n}(x_n, y_n).$$

Let us start with the lower bound, let $\varepsilon > 0$ and take a curve $\tau \subset M$ of bounded rotation from x_0 to y_0 such that

$$L_{\lambda_0}(\tau) \leq \rho_{\lambda_0}(x_0, y_0) + \varepsilon.$$

We can concatenate τ with the curves γ_n and σ_n to obtain a curve τ_n. It is clear that the τ_n converge to τ, as well as the fact that the absolute rotations of the curves τ_n are uniformly bounded, see the observation in the proof of Theorem 5.6. Hence, we can apply Lemma 5.20,

$$L_{\lambda_n}(\tau_n) \xrightarrow[n\to\infty]{} L_{\lambda_0}(\tau).$$

It then follows for all n that

$$\rho_{\lambda_n}(x_0, y_0) \leq L_{\lambda_n}(\tau_n) \xrightarrow[n\to\infty]{} L_{\lambda_0}(\tau) \leq \rho_{\lambda_0}(x_0, y_0) + \varepsilon.$$

So in the limit, since ε was arbitrary we see that

$$\limsup \rho_{\lambda_n}(x_n, y_n) \leq \rho_{\lambda_0}(x_0, y_0).$$

For the upper bound let us denote the indexes (n_k) be such that

$$\lim_{n_k\to\infty} \rho_{\lambda_{n_k}}(x_{n_k}, y_{n_k}) = \liminf_{n\to\infty} \rho_{\lambda_n}(x_n, y_n).$$

Define η_k be a curve from x_{n_k} to y_{n_k} that is shortest under $\rho_{\lambda_{n_k}}$. From Corollary 5.22 we have that the absolute rotations of the η_k are bounded by a constant $A < \infty$. It then follows from Theorem 5.9 that we can extract a subsequence from η_k that converges to a curve η_0 from x_0 to y_0. Lemma 5.20 then yields

$$L_{\lambda_{n_k}}(\eta_k) \to L_{\lambda_0}(\eta_0).$$

Since by definition we have that

$$\rho_{\lambda_0}(x_0, y_0) \leq L_{\lambda_0}(\eta_0),$$

we conclude

$$\rho_{\lambda_0}(x_0, y_0) \leq \lim_{n_k\to\infty} L_{\lambda_{n_k}}(\eta_k) = \liminf_{n\to\infty} \rho_{\lambda_n}(x_n, y_n).$$

$\square$

In order to improve the Intermediate Distances Convergence Theorem to the full Distances Convergences Theorem we would require some additional machinery to deal with points that have measure $\geq 2\pi$. This is known as canonical stretching and it allows us to compare the local behaviour of our distance with the distance of a flat cone. We do not have the space to develop these tools here, so refer the reader to (Fillastre 2023, p. 109) for a concise overview and (Reshetnyak 1960b) for complete detail.

5.4 Bounded Integral Curvature

We are now in a position to state and prove the main results of this project. The subharmonic distance that we defined without any mention of surfaces with bounded integral curvature actually give rise to the same class of surfaces.

We will of course need to work with Theorems 4.5 and 4.8. There we briefly defined the absolute turn of a curve when working over a Riemannian surface. We know from chapter 2 that we can always find isothermal coordinates. Working with our representation for λ we have the following.

Definition 5.23. Let $L = \{(u(s), v(s)) : s \in [0, l]\}$ be a curve length parametrization of a curve in a Riemannian surface where u, v are C^2. Denote by k_g the geodesic curvature. We define the turn of L by

$$\kappa_g(L) = \int_0^l k_g(s)ds.$$

We define the absolute turn of L by

$$|\kappa_g|(L) = \int_0^l |k_g(s)|ds.$$

Lemma 5.24. *Suppose $L = \{(x(s), y(s)) : s \in [a, b]\}$ is a regular curve on a Riemannian surface, where x, y are isothermal coordinates. We can represent the geodesic curvature as*

$$\kappa_g(L) = \kappa(L) - \frac{1}{2\pi} \iint_R \varphi(L, \xi)\, d\omega(\xi) - \frac{1}{2} \int_L \frac{\partial h(z)}{\partial \nu}\, d\sigma,$$

where $\kappa(L)$ is the rotation of L on a Euclidean surface(that is, we have $d\sigma^2 = dx^2 + dy^2$). By $\varphi(L, \xi)$ we mean the angle for which L is seen from the point ξ. The representation $\lambda = \lambda(\omega, h)$ gives us the measure and harmonic function in the formula.

The derivation can be found in full in (Reshetnyak 1993, p. 110). We can calculate the absolute turn of L by simply taking the total variation of $k_g(L)$. This gives us a way to deal with the bound in the statement of Theorem 4.8.

Theorem 5.25. *Let ρ_λ be a subharmonic distance on a domain M, where $\lambda(z) = \lambda(z; \omega, h)$. Then M endowed with this distance is a surface of bounded integral curvature. Moreover, all points $z \in M$ for which $\omega(\{z\}) > 2\pi$, and some of the points z such that $\omega(\{z\}) = 2\pi$, are the points at infinity of M for the distance ρ_λ.*

In preparation of the proof, we will need two preliminary results. Let us introduce the following notation. Let $\lambda(\omega)$ be defined over the plane, we define

$$\sigma_\lambda(r, z) = L_\lambda(\partial C_r(z)),$$

where $\partial C_r(z)$ is boundary of the circle of radius r centered at z.

Lemma 5.26. *Let $\lambda(z) = \lambda(z;\omega)$. Suppose z_0 is not a point at infinity for the metric ρ_λ and let $d(r)$ be the diamater of the disc $Q_r(z_0)$ endowed with ρ_λ. Then $d(r) \to 0$ as $r \to 0$.*

Proof. First, we choose a radial segment L starting at z_0. Since z_0 is not a point at infinity, the distance of L is finite and continuous. Thus, we can choose a point z_1 on L such that $L_\lambda([z_0, z_1]) < \frac{\varepsilon}{3}$. We choose a threshold radius $r_1 \leq |z_1 - z_0|$ such that for all $r < r_1$, $\sigma_\lambda(r, z_0) < \frac{\varepsilon}{3}$ (that this can be done follows by taking the contrapositive of Theorem 12.1 in (Reshetnyak 1960*b*)).

Let us now take two points x, y in the disc $Q_r(z_0)$. We define C_x and C_y to be the circles centered at z_0 that pass through x and y respectively. We can join the two points by a curve γ of bounded rotation. Beginning at x, we follow the arc of C_x until we reach L, we then move across this segment to C_y and then follow the arc to get to y. Since $\sigma_\lambda(r, z_0) < \frac{\varepsilon}{3}$ it is clear that the two arcs have length shorter than $\frac{\varepsilon}{3}$, it follows that $L_\lambda(\gamma) < \varepsilon$ and hence $\rho_\lambda(x, y) < \varepsilon$. Since x and y were chosen arbitrarily we can conclude that $d(r) \leq \varepsilon$. $\square$

As we saw before, points where $\omega(\{z\}) = 2\pi$ may be at infinity. Since we can not tell this a priori, the Distances Convergences Theorem is unable to deal with them. The following lemma will allow us to construct a sequence ω_n that avoids such points.

Lemma 5.27. *Let $\lambda(\omega, h)$ be a function over the plane, let M be a domain whose boundary is made up of a finite number of simple closed curves of bounded rotation that are pairwise non-intersecting. Then we can find a sequence of Riemannian distances ρ_{λ_n}, such that $\lambda_n = \lambda(\omega_n, h)$. We have that $\omega_n \rightharpoonup \omega$ and that ρ_{λ_n} converges uniformly on any closed set $F \subset M$. These closed sets do not share any common points with the boundary of M and also do not contain any points at infinity for ρ_λ.*

Proof. We follow the proof given in (Reshetnyak 1960*a*, p. 106) with some modifications made to explain some points in greater detail. In the interest of space some technical steps in the construction of λ_n have been ommited and we refer the reader to where these can be found. For every z we have

$$\log(\lambda(z)) = \frac{1}{\pi}\int \log\left|\frac{1}{z-\xi}\right| d\omega^+(\xi) - \frac{1}{\pi}\int \log\left|\frac{1}{z-\xi}\right| d\omega^-(\xi) + h(z).$$

We focus on

$$u(z) = \int \log\left|\frac{1}{z-\xi}\right| d\omega^+(\xi).$$

We apply a C^∞ mollifier α_h (a radial bump functions whose integral evaluates to 1). This gives us a sequence of C^∞ functions, $u_h(z) = (u * \alpha_h)(z)$ by means of convolution. Since $-u$ is subharmonic, by the mean value property we find that $u_h(z) \leq u(z)$ for all z.

The smoothed potential u_h corresponds to a smooth Lebesgue density $f_h(\zeta)$ (see (Reshetnyak 1960a, p. 107) for the construction) defined by

$$f_h(\zeta) = \int \alpha_h(\zeta - \zeta_1) d\omega^+(\zeta_1).$$

The functions f_h are C^∞ and the measures $f_h(z)dudv$ converge weakly to ω^+. We then define the approximating measures ω_n by choosing a sequence (h_n), where $h_n > 0$, $h_n \to 0$, then setting

$$\omega_n(E) = \int_E f_{h_n}(z)dudv - \omega^-(E).$$

We can now take the sequence $\lambda_n(z)$ to be given by

$$\lambda_n(z) = \lambda(z; \omega_n, h) = \exp\left(\frac{1}{\pi}u_{h_n}(z) - \frac{1}{\pi}\int \log\left|\frac{1}{z-\zeta}\right| d\omega^-(\zeta) + h(z)\right).$$

Since we have that $u_h(z) \leq u(z)$ we know that for all n it is true that $\lambda_n(z) \leq \lambda(z)$. We also claim that $\omega_n \rightharpoonup \omega$, by decomposition it is clear that it is enough to show that $\omega_n^+ \rightharpoonup \omega^+$ which is what we claimed above. Consider any bounded continuous function φ, we would like to show the following

$$\lim_{n\to\infty} \int \varphi(z) f_{h_n}(z)dz = \int \varphi(z)d\omega^+(z)$$

By substituting the definition of f_{h_n} and an application of Fubini's Theorem, we see that the left hand side would become

$$\int \left(\int \varphi(z)\alpha_{h_n}(z - \zeta_1)dz\right) d\omega^+(\zeta_1)$$

The inner integral is just the average of φ over a disc centered at ζ_1 with radius h_n. Since φ is continuous, this converges uniformly to $\varphi(\zeta_1)$. So the entire expression converges to the integral of φ against the original measure ω^+.

With this done, we now move to showing that ρ_{λ_n} converge to ρ_λ uniformly on all closed sets F defined as in the statement. This reduces to showing that if we have two convergent sequences $x_n \to x_0$ and $y_n \to y_0$, such that x_0, y_0 are in the interior of M and that none are points at infinity. Then,

$$\rho_{\lambda_n}(x_n, y_n) \xrightarrow[n\to\infty]{} \rho_{\lambda_0}(x_0, y_0).$$

If $\omega^+(\{x_0\}) < 2\pi$ and $\omega^+(\{y_0\}) < 2\pi$, then this follows immediately by the Distances Convergences Theorem. We also assume that they are not points at infinity so we can not have that the measure is $> 2\pi$. We just need to check the case when one or both points x_0, y_0 have measure ω^+ equal to 2π.

We take two discs D_r and Q_r, centered at x_0 and y_0 respectively with radius r. We denote by $d(r)$ and $\delta(r)$ the diameter of these discs under ρ_λ. Since $\lambda_n \leq \lambda$ for all n, it is clear that the diamaters of D_r and Q_r under ρ_{λ_n} are less than $d(r)$ and $\delta(r)$. Since neither point is one at infinity by assumption, we deduce from Lemma 5.26 that both $d(r)$ and $\delta(r)$ converge to 0 as $r \to \infty$. Let $\varepsilon > 0$, let $r > 0$ such that

$$d(r) + \delta(r) < \frac{\varepsilon}{3}.$$

We now choose points $\tilde{x} \in D_r$ and $\tilde{y} \in Q_r$ in the interior of M that satisfy

$$\omega^+(\{\tilde{x}\}) = 0, \omega^+(\{\tilde{y}\}) = 0.$$

By the triangle inequality we see that

$$\rho_\lambda(x_0, y_0) \leq \rho_\lambda(x_0, \tilde{x}) + \rho_\lambda(\tilde{x}, \tilde{y}) + \rho_\lambda(\tilde{y}, y_0).$$

From which we see

$$\rho_\lambda(x_0, y_0) - \rho_\lambda(\tilde{x}, \tilde{y}) \leq \rho_\lambda(x_0, \tilde{x}) + \rho_\lambda(\tilde{y}, y_0).$$

An identical application of the triangle inequality to $\rho_\lambda(\tilde{x}, \tilde{y})$ gives us

$$\rho_\lambda(\tilde{x}, \tilde{y}) - \rho_\lambda(x_0, y_0) \leq \rho_\lambda(x_0, \tilde{x}) + \rho_\lambda(\tilde{y}, y_0).$$

We have

$$|\rho_\lambda(x_0, y_0) - \rho_\lambda(\tilde{x}, \tilde{y})| \leq \rho_\lambda(x_0, \tilde{x}) + \rho_\lambda(\tilde{y}, y_0) \leq d(r) + \delta(r) < \frac{\varepsilon}{3}.$$

Since we have $x_n \to x_0$ and $y_n \to y_0$, there exists n_1, such that for $n > n_1$, $x_n \in D_r$ and $y_n \in Q_r$. For these n, an identical computation using the triangle inequality as the one above provides

$$|\rho_{\lambda_n}(x_n, y_n) - \rho_{\lambda_n}(\tilde{x}, \tilde{y})| \leq \rho_{\lambda_n}(x_n, \tilde{x}) + \rho_{\lambda_n}(y_n, \tilde{y}) \leq d(r) + \delta(r) < \frac{\varepsilon}{3}.$$

To summarise what we have so far we see that

$$\begin{aligned} |\rho_{\lambda_n}(x_n, y_n) - \rho_\lambda(x_0, y_0)| &\leq |\rho_{\lambda_n}(x_n, y_n) - \rho_{\lambda_n}(\tilde{x}, \tilde{y})| + |\rho_{\lambda_n}(\tilde{x}, \tilde{y}) - \rho_\lambda(\tilde{x}, \tilde{y})| \\ &\quad + |\rho_\lambda(\tilde{x}, \tilde{y}) - \rho_\lambda(x_0, y_0)| \\ &< \frac{2\varepsilon}{3} + |\rho_{\lambda_n}(\tilde{x}, \tilde{y}) - \rho_\lambda(\tilde{x}, \tilde{y})|. \end{aligned}$$

However, we can apply the Distances Convergence Theorem to $\tilde{x}, \tilde{y}$, which tells us that

$$\rho_{\lambda_n}(\tilde{x}, \tilde{y}) \xrightarrow[n \to \infty]{} \rho_\lambda(\tilde{x}, \tilde{y}).$$

From which we can find $n_2 > n_1$, such that for $n > n_2$ we have that the final difference is bounded above by $\frac{\varepsilon}{3}$. It follows that for these n we have

$$|\rho_{\lambda_n}(x_n, y_n) - \rho_\lambda(x_0, y_0)| < \epsilon.$$

Which concludes the proof. The reader may have noticed that here we applying the Distances Convergence Theorem to a sequence where λ_n is defined with an additional harmonic function h, which we do not have for the Distances Convergence Theorem. This is fine since we are fixing the same h for the entire sequence as well as the target λ, so no issues arise. □

We can now finally prove Theorem 5.25. The overall idea is that we would like to construct a sequence of measures with C^1 Lebesgue density that converge to the measure we use to define the subharmonic distance. Then we can simply apply the Distances Convergence Theorem. The only issue is if we encounter any points where we have measures of 2π in our approximating sequence, since then we can't apply the theorem. This is precisely why we proved Lemma 5.27.

Proof. Let ρ_λ be a subharmonic distance over M, where $\lambda(z) = \lambda(z; w, h)$. We show that for all points $z \in M$ that are not at infinity for ρ_λ, there is a neighbourhood in which we can approximate this metric by a sequence of Riemannian ones that satisy the inequality we prescribed in Theorem 4.8.

Consider a point not at infinity z_0, we denote the disc of radius r centered at z_0 by D_r. We ensure that r is small enough such that it is contained in M and also does not contain any points at infinity. For any $z \in D_r$ we have

$$\lambda(z) = \exp\left(\frac{1}{\pi}\int_{D_r} \log\left|\frac{1}{z-\xi}\right| d\omega(\xi) + h(z)\right),$$

where h is an analytic function. This additional harmonic term gets in the way and we would would ideally like to absorb it into the integral in some way. Let us consider a smaller disc D_{r_1} where we set $r_1 < r$. Over this smaller disc since we are in two-dimensions, we can represent this analytic function as a logarithmic potential (see the Localization Theorem in (Fillastre 2023, p. 34) and chapter 3 in (Folland 1995) for details),

$$h(z) = \frac{1}{\pi}\int_{C_{r_1}(z_0)} \log\left|\frac{1}{z-\xi}\right| \frac{\partial}{\partial\nu}h(\xi)d\sigma + C = \frac{1}{\pi}\int_{D_r} \log\left|\frac{1}{z-\xi}\right| d\psi(\xi) + C.$$

The measure ψ is concentrated just on the boundary of the disc and is defined for some segment E on the boundary as

$$\psi(E) = \int_E \frac{\partial}{\partial\nu}h(\xi)d\sigma$$

We take $\omega + \psi = \tilde{\omega}$ and now define

$$\lambda_0(z) = \exp\left(\frac{1}{\pi}\int_{D_r} \log\left|\frac{1}{z-\xi}\right| d\tilde{\omega}(\xi)\right).$$

Over the smaller disc, $\lambda_0(z)$ coincides with the original $\lambda(z)$. Hence, in this smaller disc, there are no points at infinity under ρ_{λ_0}. We now apply Lemma 5.27 and obtain a sequence of Riemannian distances ρ_{λ_n} that approximates λ_0. In particular, they are defined by a sequence of measures ω_n that converges weakly to $\tilde{\omega}$.

Let $\varepsilon > 0$. The uniform approximation of the distances lets us choose a measure far enough along the sequence, denote it ω_*, that provides $\lambda_*(z) = \lambda(z; \omega_*)$ with the property

$$|\rho_{\lambda_*}(x, y) - \rho_{\lambda_0}(x, y) < \frac{\varepsilon}{2}.$$

Additionally by weak convergence, we have the following bound

$$|\omega_*|(D_{r_1}) < |\tilde{\omega}|(D_{r_1}) + 1,$$

where we of course choose ω_* to be far enough along the sequence to yield both properties. The key here is since we have chosen ω_* from the sequence constructed via the lemma, the distance induced is Riemannian. So for any $z \in D_{r_1}$, we have $\omega_*(\{z\}) < 2\pi$. Actually, we have that it is 0 at every individual point, but we will not need anything better than the bound previously mentioned.

With this new measure, we can now proceed how would have wished at the beginning. Let us take a sequence ω_n of measures with C^1 Lebesgue density such that $\omega_n \rightharpoonup \omega_*$, where

$$\omega_n^+ \rightharpoonup \omega_*^+,\ \omega_n^- \rightharpoonup \omega_*^-.$$

We can now apply the Distances Convergence Theorem to the sequence of distances ρ_{λ_n} given by the measures ω_n. We repeat the same as above, we see that

$$|\rho_{\lambda_n}(x, y) - \rho_{\lambda_0}(x, y)| < \frac{\varepsilon}{2},$$

for a sufficiently large n. By weak convergence and ensuring the previous n was chosen large enough we also have

$$|\omega_n|(D_{r_1}) < |\tilde{\omega}|(D_r) + 1.$$

It now follows by the triangle inequality that

$$|\rho_{\lambda_n}(x, y) - \rho_{\lambda_0}(x, y)| < \varepsilon.$$

So we have shown that we can approximate by a sequence of Riemannian ρ_{λ_n} and additionally that $|\omega_{\lambda_n}|$ is bounded. It remains to show that the absolute turn is also bounded.

To estimate the absolute turn of the boundary of the disc ∂D_{r_1} under ρ_{λ_n} we use Lemma 5.24. Importantly, we have that $\lambda_n(z) = \lambda(z; \omega_n)$, which means we can simplify the formula since we do not have a harmonic term. Let θ be the angle for which the curve of circle ∂D_{r_1} under ρ_{λ_n} is seen from the center. We have that the turn is given by

$$\kappa_{\lambda_n}(\theta) = \theta - \frac{1}{2\pi} \int_{D_{r_1}} \varphi(\theta, z) d\omega_n(z).$$

We can now estimate the absolute turn of the entire boundary by turning an angle of 2π.

$$\begin{aligned}
|\kappa_{\lambda_n}|(\partial D_{r_1}) = \bigvee_0^{2\pi} \kappa_{\lambda_n} &\leq 2\pi + \frac{1}{2\pi} \int_{Q_{r_1}} \bigvee_0^{2\pi} \varphi(\theta, z) d|\omega_n|(z) \\
&\leq 2\pi + \int_{D_{r_1}} d|\omega_n|(z) \\
&< 2\pi + |\tilde{\omega}|(D_{r_1}) + 1.
\end{aligned}$$

□

Before we begin prove the other direction, we introduce another way to define a surface of bounded intgeral curvature.

Definition 5.28. We say that a sequence of metric spaces (R_n, ρ_n) converges to (R, ρ_R) if for all n there exists a homeomorphism $\varphi_n : R \to R_n$ such that

$$\rho_n(\varphi_n(x), \varphi(y)) \xrightarrow[n\to\infty]{} \rho_R(x, y),$$

uniformly on $R \times R$ where $x, y \in R$.

Instead of approximating by a sequence of distances over the same neighbourhood like we did in chapter 4, we will instead suppose that for any point x in our space, there exists a neighbourhood U. We then have a sequence of Riemannian surfaces with boundary (U_n) that converge to U. This way may be easier to visualise, for example, picture a sequence where the point is smoothed out that gets closer to a cone point.

To prove that we can always introduce isothermal coordinates on a surface of bounded integral curvature we are going to use the fact we can find them on the approximating surfaces.

Theorem 5.29. *Any domain in a surface of bounded integral curvature whose closure is homeomorphic to a disc is isometric to some domain in the plane endowed with a subharmonic distance. In particular, we can introduce an isothermal coordinate system. That is:*

$$ds^2 = \lambda(z)(dx^2 + dy^2), \ \lambda(z) = \lambda(z; \omega, h).$$

Proof. (Reshetnyak 1960*a*, p. 112) Let R be a domain in a surface of bounded integral curvature homeomorphic to a closed disc. Let $R_1, \dots, R_n, \dots$ be domains homeomorphic to closed discs in a Riemannian surfaces which converge to R. From chapter 4, the absolute curvature of the R_n as well as the absolute turns of the boundaries are uniformly bounded. Consider a point p in the interior of R and let $p_n = \varphi_n(p)$ be points in R_n such that $p_n \to p$. The uniform convergence of the distances ensures that p_n is always a positive distance away from the boundary of R_n. We introduce isothermal coordinates (x, y) in R_n such that $D = \{x^2 + y^2 \leq 1\}$ is the domain of coordinate change. We choose that the point p_n has coordinates $x, y = 0$. In these coordinates the metric takes the form

$$ds^2 = \lambda_n(z)(dx^2 + dy^2),$$

where $\lambda_n = \lambda_n(\omega_n, h_n)$. We can deal with the harmonic function in the same way we did in the previous theorem, see the Localization Theorem in (Fillastre 2023, p. 34) and chapter 3 in (Folland 1995) for details. We have

$$h_n(z) = \frac{1}{\pi} \int_D \log \left| \frac{1}{z - \xi} \right| d\psi_n(\xi) + c_n$$

where ψ_n is a measure concentrated on the boundary of the disc D. It can be shown that the variations of ψ_n are uniformly bounded, in the interest of space we refer the reader to (Reshetnyak 1960a, p. 111-112) for details. By construction, the measures ω_n are uniformly bounded. So we consider

$$\tilde{\omega_n} = \omega_n + \psi_n,$$

whose variations are also uniformly bounded. It follows by the Banach-Alaoglu Theorem that we can find a subsequence that converges weakly. Let us suppose that $(\tilde{\omega}_n)$ is already this subsequence, thus $\tilde{\omega}_n \rightharpoonup \tilde{\omega}_0$.

Let us now consider a subsequence (c_{n_m}) that converges to a limit c_0, which we do not know is finite for the moment. Suppose the measures $\tilde{\omega}_n^+ \rightharpoonup \tilde{\omega}_0^+$ and denote by $\{z_1, \dots, z_p\}$ all the points on the discs such that $\tilde{\omega}_0^+(\{z_i\}) \geq 2\pi$. We consider a new distance induced by

$$\tilde{\lambda}_{n_m}(z) = e^{-c_{n_m}} \lambda_{n_m}(z).$$

By the Distances Convergence Theorem, the distances $\rho_{\tilde{\lambda}_{n_m}}$ converge to $\rho_{\tilde{\lambda}_0}$ uniformly on every closed set without the points z_i. The distance $\rho_{\tilde{\lambda}_0}$ is defined by $\tilde{\lambda}_0(z) = \lambda(z; \tilde{\omega}_0)$. We would like to show that c_0 is finite, let us begin by showing that $c_0 < \infty$. Recall that our sequence of Riemannian surfaces converges to R, by the uniform convergence of the distances, we have that the diameters of the domains R_n are uniformly bounded. Consider two points $x, y \notin \{z_1, \dots, z_p\}$ such that $\rho_{\tilde{\lambda}_0}(x, y) > 0$. The distances $\rho_{\tilde{\lambda}_{n_m}}(x, y)$ converge to $\rho_{\tilde{\lambda}_0}(x, y)$. But now see that the distance between these two points with respect to $\rho_{\lambda_{n_m}}$ must be $e^{c_{n_m}} \rho_{\tilde{\lambda}_{n_m}}(x, y)$. If $c_{n_m} \to \infty$ then we can not have that the diameters of R_{n_m} are bounded above.

To see that we also have $c_0 > -\infty$ we consider a triangle T whose vertices lie on the boundary of the disc D. Suppose that $\{z_1, \ldots, z_p\} \notin \partial T$ so that the perimeter of T under $\rho_{\tilde{\lambda}_0}$ is bounded. Suppose for a contradiction that we did not have $c_0 > -\infty$. We have that $L_{\tilde{\lambda}_{n_m}}(\partial T) \to L_{\tilde{\lambda}_0}(\partial T)$ by Lemma 5.20. The exponential term in λ_{n_m} now gives us that $L_{\lambda_{n_m}} \to 0$. The uniform convergence of distances on R_{n_m} to that of R now tells us that the lengths of the three boundary curves of our triangle T vanish. Forcing the three vertices to coincide at a common point, which is a contradiction due to how we have defined R.

To simplify the notation, let us re index our sequence of distances from n_m to just m. That is, in the disc D, we have a sequence of distances ρ_{λ_m} converging uniformly to ρ_{λ_0} on every closed set without the points at infinity $\{z_1, \ldots, z_p\}$. Our distances are given by $\lambda_m(z) = \lambda(z; \tilde{\omega}_m, c_m)$.

We know that for each m, by Definition 5.28, that there exists homeomorphisms $\varphi_m : R \to R_m$. If we denote the mapping of the domain R_m onto the unit disc D by α_m then we can define a mapping $\beta_m : R \to D$ constructed as

$$\beta_m = \alpha_m \circ \varphi_m.$$

Let $f_m = \varphi_m^{-1} \circ \alpha_m^{-1}$ be the inverse mapping of β_m, mapping Q onto R. Let us now construct a new distance

$$\tilde{\rho}_m(x, y) = \rho_R(f_m(x), f_m(y)),$$

which we claim is the uniform limit in D of the distances ρ_{λ_m}. This follows from

$$\begin{aligned}
\rho_{\lambda_m}(x, y) - \tilde{\rho}_m(x, y) &= \rho_{R_m}\left(\alpha_m^{-1}(x), \alpha_m^{-1}(y)\right) - \rho_R(f_m(x), f_m(y)) \\
&= \rho_{R_m}\left(\varphi_m(f_m(x)), \varphi_m(f_m(y))\right) - \rho_R(f_m(x), f_m(y)),
\end{aligned}$$

where the right hand side converges unifomrly to 0 in Q by Definition 5.28. But we already know that ρ_{λ_m} converges to ρ_{λ_0} uniformly on all closed sets without the points $\{z_1, \ldots, z_p\}$. The triangle inequality then tells us that $\tilde{\rho}_m$ must also converge to ρ_{λ_0} uniformly on every closed set without the points $\{z_1, \ldots, z_p\}$. Of course, we would like to show that this convergence occurs on the whole disc D.

We first claim that none of the points $\{z_1, \ldots, z_p\}$ are at infinity for the distance ρ_{λ_0}. Let us suppose for a contradiction that z_i is such a point. For every $N \in \mathbb{N}$ can find two concentric circles around z_i such that none of the remaining points in $\{z_1, \ldots, z_p\}$ lie within them and the distance between the two circles for ρ_{λ_0} is larger than N. It follows that for sufficiently large m, the distance between the two circles for ρ_{λ_m} will also be greater than N. Since the disc D endowed with ρ_{λ_m} is isometric to R_m, we see that for these large m, the diameters of R_m are larger than N. Which contradicts the fact they are uniformly bounded.

The idea now is to isolate the problem points with small discs of size ε to ensure that whilst we can not deal with the point itself, we have an ε control on the small area around it and so can still approximate. To this end, since we have shown that the distance ρ_{λ_0} does not have any points at infinity, there exists a function $\delta(\varepsilon)$. Such that for any domain in D whose boundary is a simple closed curve with boundary bounded by $\delta(\varepsilon)$ the diameter with respect to ρ_{λ_0} of the domain itself does not exceed ε. We can define for $\tilde{\rho}_m$ the functions $\delta(\varepsilon)$ in the same way. Note that since all the $\tilde{\rho}_m$ are isometric, these functions are the same for all m. Let us take $\delta_0(\varepsilon)$ to be the minimum of $\delta(\varepsilon)$ and $\delta_m(\varepsilon)$.

We now construct our discs, for any point $z_i \in \{z_1, \ldots, z_p\}$ let Q_i be an open disc centered at z_i. We choose these discs such that they do not share any common points with each other and the diameter of ∂Q_i under ρ_{λ_0} is bounded above by $\delta_0(\varepsilon)$. On the boundary of each disc, we have $\tilde{\rho}_m \to \rho_{\lambda_0}$ uniformly. This implies that we can find suffiently large m_1 such that for all $m > m_1$ the diameter of the boundary of our disc under $\tilde{\rho}_m$ will be less than $\delta_0(\varepsilon)$. For these m it follows that the diameter of the disc itself with respect to $\tilde{\rho}_m$ will be less that ε.

Let us denote the set with these discs removed by

$$P_\varepsilon = D \setminus \bigcup_{i=1}^{p} Q_i.$$

We now consider any two points x, y in D and define $\tilde{x}, \tilde{y}$ to be the closest points on the set P_ε to them. For $m > m_1$ we can apply the triangle inequality (in the same way as we did explicitely in the proof of Lemma 5.27) to see

$$\begin{aligned}|\rho_{\lambda_0}(x,y) - \rho_{\lambda_0}(\tilde{x},\tilde{y})| &\leq \rho_{\lambda_0}(\tilde{x},x) + \rho_{\lambda_0}(\tilde{y},y) < \frac{2\varepsilon}{5},\\ |\tilde{\rho}_m(x,y) - \tilde{\rho}_m(\tilde{x},\tilde{y})| &\leq \tilde{\rho}_m(\tilde{x},x) + \tilde{\rho}_m(\tilde{y},y) < \frac{2\varepsilon}{5}.\end{aligned}$$

Recall that we know that $\tilde{\rho}_m$ converges uniformly to ρ_{λ_0} on any closed set that does not contain the problem points z_i. By how we have defined P_ε we know that it is closed, so it follows that we can find m_2 such that for every $m > m_2$ we have

$$|\tilde{\rho}_m(x,y) - \rho_{\lambda_0}(x,y)| < \frac{\varepsilon}{5}$$

for all $x, y \in P_\varepsilon$. It then follows that if we take $m > \max\{m_1, m_2\}$ that we have

$$\begin{aligned}|\rho_{\lambda_0}(x,y) - \tilde{\rho}_m(x,y)| &\leq |\rho_{\lambda_0}(x,y) - \rho_{\lambda_0}(\tilde{x},\tilde{y})| + |\rho_{\lambda_0}(\tilde{x},\tilde{y}) - \tilde{\rho}_m(\tilde{x},\tilde{y})|\\ &\quad + |\tilde{\rho}_m(\tilde{x},\tilde{y}) - \tilde{\rho}_m(x,y)|\\ &< \varepsilon,\end{aligned}$$

for arbitrary x, y chosen in D.

We would like to apply Arzelà–Ascoli to sequence of mappings f_m. That they are uniformly bounded is clear since the diameter of R is bounded. We now claim that the f_m from D endowed with the distance ρ_{λ_0} onto R are also equicontinuous. We have just proved that $\tilde{\rho}_m \to \rho_{\lambda_0}$ uniformly on the entire disc D. So if we choose $\varepsilon > 0$, we have that for any $m > \max\{m_1, m_2\}$,

$$|\rho_{\lambda_0}(x, y) - \tilde{\rho}_m(x, y)| < \frac{\varepsilon}{2}.$$

Using the triangle inequality gives us the following upper bound on the distance in R,

$$\rho_R(f_m(x), f_m(y)) = \tilde{\rho}_m(x, y) \leq \rho_{\lambda_0}(x, y) + \frac{\varepsilon}{2}.$$

If we choose $x, y \in D$ such that

$$\rho_{\lambda_0}(x, y) < \frac{\varepsilon}{2},$$

then for all $m > \max\{m_1, m_2\}$ it follows that

$$\rho_R(f_m(x), f_m(y)) < \varepsilon.$$

Hence, we can choose a subsequence f_{m_k} that converges uniformly to some continuous mapping $f_0 : D \to R$. We finally claim that the mapping f_0 is isometric, to see this we have

$$\rho_R(f_0(x), f_0(y)) = \lim_{k\to\infty} \rho_R(f_{m_k}(x), f_{m_k}(y)) = \lim_{k\to\infty} \tilde{\rho}_{m_k}(x, y) = \rho_{\lambda_0}(x, y).$$

Thus, we have shown that the disc D with the metric ρ_{λ_0} is isometric to R. $\square$

6 Further Remarks

We have shown that the frameworks built by Alexandrov and Reshetynak give rise to the same class of surfaces. In this final chapter we briefly discuss how this theory can be applied as well as some later modifications made to the theory presented already.

In the first chapter we showed that under certain conditions we could view Riemannian surfaces as Riemann surfaces, using the existence of local isothermal coordinates as a bridge. Using the theory we have developed we have the following results.

Theorem 6.1. *Let $\varphi : U \to \mathbb{C}$, $\psi : V \to \mathbb{C}$ be isothermal coordinate systems in a two-dimensional manifold of bounded curvature. We assume that the set $U \cap V$ is not empty, and let $G = \varphi(U \cap V)$, $H = \psi(U \cap V)$. Then G and H are open sets in $\mathbb{C}$ and the function $\theta(z) = \psi[\varphi^{-1}(z)]$ is a conformal map of G onto H. If $ds^2 = \lambda(z)|dz|^2$ is the metric corresponding to the coordinate system φ, and $d\sigma^2 = \mu(w)|dw|^2$ is the metric* *corresponding to the coordinate system ψ, then for all $z \in G$ we have*

$$\lambda(z) = \mu[\theta(z)]|\theta'(z)|^2.$$

See (Huber 2023, p. 355). Specifically, this tells us that if M is a surface of bounded integral curvature, then M can be viewed as a Riemann surface where the local isothermal coordinate systems form a set of charts. In particular, if $\lambda(z)|dz|^2$ and $\mu(w)|dw|^2$ are the different differential quadratic forms that correspond to two isothermal coordinate systems, then the functions λ and μ are linked by the equation provided in Theorem 6.1. Thus, on the Riemann surface there is a defined quadratic differential σ.

Huber also proved that the converse is true.

Theorem 6.2. *Let M be a two dimensional Riemann surface. Assume that on M there is a specified quadratic differential σ such that for any local coordinate system the coefficient is defined by $\lambda(z) = \lambda(z; \omega, h)$. If M is connected, then we define on it an intrinsic distance ρ_σ such that for any coordinate system $\phi : U \to \mathbb{C}$ any point $p \in U$ has a neighbourhood $V \subset U$ with the property that for any $x, y \in V$ we have*

$$\rho_\sigma(x, y) = \rho_\lambda(\phi(x), \phi(y)).$$

Where ρ_λ is the distance defined on the domain $G = \phi(U)$ by the metric $ds^2 = \lambda(z)|dz|^2$, which is the representation of σ in the coordinate system ϕ. The distance ρ_σ satisfying these conditions is unique and (M, ρ_σ) is a surface of bounded integral curvature.

This theorem provides us with a way to move away from the theory of surfaces of bounded integral curvature and apply a wider range of tools. Allowing us to study in the large the structure of surfaces of bounded integral curvature. Examples of how this can be applied in practise can be found in (Huber 1958).

When defining our subharmonic distance, we specifically restricted the infimum to curves of bounded rotation. The reader may wonder if taking it over other classes may still lead to the same class of surfaces.

In the case where for every point $z \in M$ we have $\omega^+(\{z\}) < 2\pi$ we obtain the same distance by restricting to any set of curves that contain the piecewise linear ones. In particular, we could work with piecewise C^∞ curves. This is discussed further in (Fillastre 2023, p. 106). Over a more general surface we can not assert this anymore however we can widen the class instead. We can work with just rectifiable curves instead of just those of bounded rotation. This was proved by Reshetynak a few years after the work we have highlighted in this project, see (Reshetnyak 1963).

While the framework developed in Chapter 5 successfully connects subharmonic distances to surfaces of bounded integral curvature, we mention that this class does not accommodate certain surfaces we may wish to study. The removal of points at infinity means the surfaces we construct can not possess features such as hyperbolic cusps. Geometrically, it is a point at infinity where the space stretches out endlessly into the shape of a funnel or horn, becoming infinitely narrow without ever actually closing up or coming to an end. In the view of Definition 5.28, we can perhaps think of these as being approximated by surfaces of bounded integral curvature where for each approximating surface the tip of the cusp is rounded off. However, the limit space is not included in our current framework. This is posed as Problem 9.3 in a modern survey of the theory (Troyanov 2023, p. 30).

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